sity; "The relation between the size of the energy atom and its physiological effect" by W. T. Bovie; "Some aspects of celestial evolution" by Professor E. B. Frost of Yerkes Observatory; "The laws of racing fatigue in men and horses" by Professor A. E. Kennelly of Harvard University; "The alleged sins of science" by Dr. R. A. Millikan, president of the American Association for 1929; and a stereopticon lecture on the new "Max Adler Planetarium of Chicago" by Professor Philip Fox, recently of Northwestern University and now Director of the Planetarium. On Tuesday afternoon the seventh Josiah Willard Gibbs lecture was given by Professor Irving Fisher of Yale University on "The application of mathematics to the social sciences." This address was given under the joint auspices of the American Association and the American Mathematical Society; an audience of over four hundred indicated the general interest in the subject and the speaker.

The scientific exhibition was held in the large registration hall of the Shrine Temple and included the exhibits of numerous publishers of college and university texts and of makers of scientific apparatus. Of especial interest to mathematicians was an exhibit of old books and incunabula on mathematics, astronomy and physics from the library of Professor J. S. Turner of Iowa State College shown at Des Moines Public Library; this was made possible through the generosity of Professor Turner and the cooperation of the department of mathematics of Iowa State College.

The groups of mathematicians stayed for the most part in Hotel Fort Des Moines, and the mathematics meetings were all held in suitable rooms in this hotel. As a consequence there was unusual opportunity for the informal social aspects of the meetings. About one hundred thirty attended the joint dinner on Tuesday evening in the Oak Room, Professor W. H. Bussey, editor of the MONTHLY, acting as toastmaster and calling on various representatives of the two organizations. Professor Rietz welcomed the guests to Iowa and spoke of the importance of the meetings for that region. Professor J. W. Young, coming from New England, spoke as president of the Association and mentioned the difficulty experienced by mathematicians in keeping in touch with each other in our country-wide dispersion and the necessity, in view of our extensive and ever-increasing literature, of obtaining publication funds on a large scale. Miss Jewell Hughes, in a speech replete with wit, depicted some of the lighter, even if logical, experiences of those mathematically inclined. Professor Hedrick, coming from the Pacific Coast, spoke as president of the Society, agreeing with Professor Young in the hope that sooner or later we may be free from the present burden of seeking adequate finances for the Society. He expanded his remarks made at the close of the Gibbs lecture by saying that we must not limit ourselves, in our definition of the scope of mathematics, to purely formal matters but must include all those fields which can be treated by the instrumentalities of mathematics, and we must include such men as Gibbs, who was primarily a physicist, in the list of those doing essentially mathematical work.

The Des Moines meetings are noteworthy by reason of the hospitality shown by the citizens of Des Moines and that region. A cordial welcome was evident on every hand, the Des Moines Club entertained groups of scientists each noonday, and the authorities of the State University of Iowa and of Iowa State College made it possible for the visitors to see the grounds, buildings and laboratory facilities of these institutions. These and similar services were recognized in a resolution, adopted at the joint dinner by a rising vote, expressing the appreciation of the visitors for the admirable efficiency of the arrangements and the generosity of the entertainment, in particular expressing to Mr. Hunter of the Equitable Life Insurance Company of Iowa, to President Morehouse and Professor Neff of Drake University, to Iowa State College and the State University of Iowa, our cordial thanks and the assurance that the meetings will long be a cherished memory.

The American Mathematical Society held sessions in Hotel Fort Des Moines on Monday and Tuesday mornings, partly in the form of simultaneous sections, for the reading of papers, one section Tuesday morning being a joint session with Section K (Economics and Social Sciences). It held a joint session Monday afternoon with Section A (Mathematics) of the American Association, at which Professor R. C. Archibald gave his retiring address as Vice-President of Section A on "Mathematics before the Greeks" and Professor O. D. Kellogg gave an address on "An unsolved problem of uniqueness in potential theory" on invitation of the Society and Section A. The Gibbs lecture for 1929 delivered by Professor Irving Fisher, on invitation of the Society, has already been mentioned.

The program of the Mathematical Association consisted of a joint session with the Society on Tuesday afternoon and two sessions on Wednesday. The program was prepared by a committee consisting of Professors Arnold Dresden, L. M. Graves, W. L. Hart, H. W. Kuhn, and E. W. Chittenden, Chairman.

Joint Session of the Association with the Society

"Linear inequalities and some related properties of functions" by Professor L. L. Dines, University of Saskatchewan, by invitation of the two organizations.

This interesting and instructive address will appear in the *Bulletin* of the American Mathematical Society, since it is somewhat more suited to that publication.

First Session of the Association

1. "The geometric representation of groups" by Professor R. P. Baker, University of Iowa.

2. "Aspects of the theory of linear associative algebras of infinite order" by Professor M. H. Ingraham, University of Wisconsin.

3. "The fourth Carus monograph" by Professor J. W. Young, Dartmouth College.

1. The paper gave a description of five classes of representations of a group with illustrations on lantern slides. These are (1) Schraffirte diagrams, (2) Cayley diagrams with independent generators, (3) General Cayley diagrams, (4) Geometric configurations with a group of transformations, (5) Representations of the substitution letters by geometric objects.

In field (2) Maschke (American Journal, vol. 18) enumerated the Cayley diagrams on the sphere; these are all half regular.

In carrying on this work to the projective plane and anchor ring it is practically necessary to postulate half-regularity to prevent being overwhelmed with examples mostly bizarre. For the projective plane only three such diagrams are possible, for $G_{12\cdot 4}$, $G_{24\cdot 4}$, $G_{60\cdot 5}$. For the anchor ring 68 sets exist. Four of these are unique groups of order 16. The others have one, two, or three arbitrary integers in the order. They are based on 7 nets of the half-regular type. Three other nets are possible but lead to an extra generator and the situation explored by Dyck for groups of genus one. These groups can be described as having an Abelian self conjugate subgroup with two independent generators. The quotient group, which is cyclic for groups of genus one, may be E, C_2, C_3, C_4, G_4, G_6, C_6, $G_{8\cdot 4}$, $G_{12\cdot 5}$.

Looking at the repeated development of the anchor ring in the plane as a homogeneous assemblage the elements may be called 'atoms' and the sets connected by generators of the quotient group 'molecules.'

The generators of the Abelian subgroup can be called 'translations' but 'rotations' is hardly applicable to the operations of the quotient group.

Incidentally the work led to a complete solution of the space-filling problem for the plane, ten types dual to the ten nets of this problem.

2. The usual postulates for a linear associative algebra may be generalized in several directions. Interesting results have been obtained by omitting the associative law. This paper deals with associative algebras which do not have a finite base in terms of which all elements can be expressed as linear combinations with scalar coefficients. Wedderburn, in an article[1] entitled *Algebras which do not possess a finite base*, has developed a theory which considers not only finite sums but such sums as sums of series and integrals. J. W. Young has given a very interesting point of departure for the theory of algebras from the group standpoint in two papers *On the partitions of a group and the resultant classifications* (Bulletin American Mathematical Society, vol. 33, pp. 453–461), *A new foundation for general algebra* (Annals of Mathematics, vol. 29, pp. 147–60. This paper chiefly discusses generalizations which arise from assuming that aggregates can be well ordered, and in certain portions that the cardinal number of the algebra is an "aleph" number. The generalizations of the theory of linear sets, reducibility, matrix representations and nilpotent algebras are discussed.

[1] Transactions of the American Mathematical Society, vol. 26 (1924), pp. 395–426.

3. Dr. J. S. Georges who was to have given a paper on "Some social aspects of mathematics" was unable to be present because of illness in the family. Professor Young was therefore asked by the program committee to give a description of the forthcoming Carus monograph.

The subject matter of projective geometry can be studied with only a small amount of previous mathematical knowledge and hence this monograph can be thought of as adapted to gaining recruits for mathematics. Approximately the first half is of rather simple character and is purely synthetic. The development of projective geometry out of elementary geometry through metric considerations has its drawbacks, for it does not show how projective properties can be established without using metric geometry.

On the other hand such a monograph as this should show the relation of projective to metric geometry; hence after giving the general properties of conic sections, there is a study of the specializations necessary for exhibiting the various special geometries.

Professor Young has depended on an intuitive rather than a postulational basis; for example, he has tacitly assumed the postulates of alignment.

In the latter half he lays the foundation for an analytic treatment on a purely projective basis, and he brings out the definitions of the various geometries in terms of groups of transformations.

Professor Slaught emphasized the importance of having the Association members suggest in written form desirable subjects for future monographs and spoke of the ideals which have always been kept in the minds of the committee in carrying on the development of the monographs. He announced that the Young monograph was then in page proof and might be expected to be out late in March.

Second Session of the Association

1. "Appreciation of form as an index of mathematical ability" by Professor E. B. Lytle, University of Illinois.

2. "The laws of rolling motion and the complete elimination of sliding friction in roller bearings" by Professor W. H. Roever, Washington University.

3. "Mathematics and the problem of ore location" by Professor Warren Weaver, University of Wisconsin.

1. An analysis of reflective thinking shows at least three stages: first, there is a problem, a question, or a difficulty; secondly, there must be suggested possible answers, possible methods of procedure, or possible investigations to make; and thirdly, all suggestions produced must be carefully tested to determine the best one. Often the second stage is not sufficiently emphasized, for how to get suggestions when perplexed is a common difficulty in reflective thinking. When thinking with students or before students, teachers might be more helpful if they were more careful to exhibit what are the sources of their ideas, what are their cues to action, what suggests the transformation or operation they carry

out. One very fertile suggestor in mathematics is the notion of algebraic form. Transformations are suggested by the desire to get certain forms which are rich in meanings; such transformations generally seem to be mere symbol juggling to one unappreciative of forms. Ability to get information and cues to action from forms is so necessary in mathematical work that this ability may be taken as an index of mathematical ability.

2. The paper was introduced with a statement of the Poncelet theorem that the instantaneous state of velocities of a rigid body in motion is that due to a certain ruled surface of the body moving on a ruled surface fixed in space in such a way that the former surface is tangent to the latter all along a common generator, on which the body rolls and slides. It was then shown that two rigid bodies in contact with each other may slide, roll, or pivot at the contact. If, in particular, the sliding is nil, the rolling and pivoting are tangential and normal components of the instantaneous velocity vector, which now passes through the contact. Owing to the fact that sliding friction is greater than rolling friction, journal bearings have been replaced by roller (including ball) bearings. In early forms of the latter no provision was made for separating consecutive rollers, and therefore sliding occurred at the contacts of such rollers, thus counteracting to some extent the advantage of rolling at other points of the bearing. To obviate this defect, devices called "cages" were used to separate rollers, but these introduced sliding friction at the points where they came into contact with the rollers which they separated. Pictures were shown of several types of such devices. However, it was shown by simple apparatus that it is possible to separate rollers without having sliding motion at any of the contacts. Pictures were shown of a bearing in which use was made of this principle. Since balls must travel in channels to be properly guided, pivoting, as well as rolling, occurs where balls are used, but sliding friction can be entirely eliminated.

3. This paper opens with introductory remarks concerning earth physics, and the relations of this subject to atomic and astro-physics. It is then pointed out that any geo-physical prospecting method consists essentially of a sending device which dispatches messengers down into the earth, and a receiving device which decodes the messages when part of the messengers return to the surface. The magnetic, electrical, electro-magnetic, gravitational, and seismic methods are then very briefly discussed. The second main division of the paper considers the mathematical problems presented by these five methods. The central unity of the problems, from a mathematical point of view, is emphasized by pointing out that all of the methods are governed by the same general differential equation, and are subject to boundary conditions of the same type. Seven general reasons are advanced to indicate why the problems of geophysical prospecting are new problems, even though they involve the application of classical physical theories, and classical mathematical methods. Various problems which have been satisfactorily treated are mentioned, as well as a considerable number of problems which have not been solved. Indication is

given of some investigations of pure mathematics, whose successful treatment would contribute greatly to advances in this new art.

This paper will appear in full in the April issue of this MONTHLY.

MEETINGS OF THE BOARD OF TRUSTEES

Eleven members of the Board of Trustees were present at the sessions.

The following fourteen persons and one institution were elected to membership on applications duly certified:

To Individual Membership

F. R. BAMFORTH, PH.D. (Chicago). Asst. Prof., Cornell Univ., Ithaca, N. Y.

R. H. CAMERON, A.B. (Cornell). Instr., Cornell Univ., Ithaca, N. Y.

JOSÉ ISAAC CORRAL, Ing. de Minas. Director, Montes y Minas, Republic of Cuba, Havana, Cuba.

NAT EDMONSON, JR., PH.D. (Rice). Acting Asst. Prof., Texas Tech. Coll., Lubbock, Tex.

ARTHUR HAAS, A.B. (C.C.N.Y.), B.S. (C.U.), LL.B (N.Y.U). Chairman, Dept. of Math., Thomas Jefferson High School, Brooklyn, N. Y.

ELIZABETH L. HARRIS, A.B. (Washington Univ.). Computer, Southwestern Bell Telephone Co., Kirkwood, Mo.

F. T. H'DOUBLER, M.D., PH.D. (Wisconsin). Springfield, Mo.

G. H. MURRAY, B.S. in Bus. Admin. (Missouri). 2934 Victor St., Kansas City, Mo.

F. P. NASH, JR., A.M. (Columbia). Head of Dept., Groton School, Groton, Mass.

H. T. H. PIAGGIO, M.A., D.Sc. Prof., Head of Dept., University College, Nottingham, England.

H. A. POOL, A.M. (Minnesota). Instr., State School of Mines, Rapid City, S. D.

MARY B. RUMSEY, A.M. (Illinois). Instr., Knox Coll., Galesburg, Ill.

W. O. SHRINER, PH.D. (Michigan). Head of Dept., State Teachers Coll., Terre Haute, Ind.

R. F. WICK, B.S. in E.E. (A.&M. Coll. of Texas). Asst., E.E. Dept., A.&M. Coll. of Texas, College Station, Tex.

To Institutional Membership

ST. BONAVENTURE'S COLLEGE, St. Bonaventure, N. Y.

The financial statement for the year 1929 was presented by the Secretary-Treasurer. It was accepted and approved by the Trustees, subject to an inspection by a committee of the Trustees; this examination was made on Wednesday by Professors Archibald, Slaught and Young and the report approved.

It was voted to authorize the Finance Committee to invest further current funds for the General Endowment as the surplus funds warrant. (As a consequence of this action, three thousand dollars have been invested in trust funds in January 1930.)

It was voted to approve the printing of a catalog of the library of the Association; to thank Professor Slaught for his effective work in connection with the Carus monographs, and the Carus family for their continued generosity; to express our interest in a proposed summer school for those who teach mathematics to engineering students to be held under the auspices of the Society for the Promotion of Engineering Education in connection with the Minnesota Summer Meeting in 1931; to express our interest and promise

our cooperation in the reestablished International Commission on the Teaching of Mathematics; and to express our interest in the plans for the possible establishment of a journal in applied mathematics.

The Trustees ratified the action of President Young in appointing H. S. Vandiver and Mrs. Anna Pell Wheeler as representatives of the Association on the Editorial Board of the Annals of Mathematics for a three-year term beginning January 1930. Because the work of the standing Committee on Relations with the Annals had been completed, this committee was discontinued.

Expressions of hearty thanks to Doctor A. B. Chace for his generous gift of several hundred sets of the Rhind Mathematical Papyrus to the Association, and to Professor Archibald for his great service involved in his manifold work on the Papyrus were adopted, these expressions of appreciation to be presented to them personally in suitable form.

After extensive consideration by correspondence and at the meetings at Boulder and Des Moines, the Trustees voted to approve and to put into operation a plan for the nomination of officers whereby a nominating committee will suggest five names of persons eligible and suitable for President, ten for Vice-President, and twenty for Trustees, when the blanks for the preliminary ballots are sent out, it being understood that this is purely advisory and is given in the interest of avoiding a scattering of ineffective votes, and not to limit the free choice of members. This plan involves no change in the By-Laws.

The Chauvenet Prize

The committee on the second award of the Chauvenet Prize recommended that the award be made to Professor T. H. Hildebrandt of the University of Michigan for his paper on "The Borel theorem and its generalizations" published in the *Bulletin* of the American Mathematical Society, volume 32(1926), pages 423–474. Professors A. J. Kempner and D. R. Curtiss for the committee stated that "the paper presents in clear and elementary fashion, and with adequate references to literature, the development of a rather broad range of ideas and results connected with the Borel theorem. It gives the reader a compact picture not easily obtained by independent reading of the scattered literature on the subject." The Trustees adopted the recommendation and the award was announced at the annual business meeting. The prize, $100 in cash, has since been conferred.

Annual Business Meeting of the Association

The Secretary-Treasurer announced the names of those elected to membership. He reported also the deaths of the following members:

J. A. Ankebrandt, Professor of mathematics, Western State College, (May 25, 1929).

C. A. Epperson, Teacher of mathematics, Kansas City, Mo., (November 5, 1928).

Sister Blanche Marie Maskell, Teacher, College of St. Elizabeth, (December 1928).

R. M. MATHEWS, Associate professor of mathematics, West Virginia University, (October 20, 1929).

A. W. PRATER, Assistant, University of California at Los Angeles, (April 16, 1929).

FRED REUSSER, Head of department of mathematics, Buena Vista College, (October 11, 1929).

E. D. ROE, JR., Professor of mathematics, Syracuse University, (December 11, 1929).

W. H. SHERK, Professor of mathematics, University of Buffalo, (January 1929).

SARAH E. SMITH, Professor of mathematics, Mount Holyoke College, (November 18, 1929).

The election of officers for the year 1930 resulted in the following, as reported by the tellers, Professors U. G. Mitchell and W. J. Risley:

For Vice-Presidents: E. T. Bell, 348 votes; Tomlinson Fort, 195 votes; W. C. Graustein, 245 votes; E. B. Stouffer, 210 votes.

For additional members of the Board of Trustees, to serve until January 1933: A. A. Bennett, 340 votes; H. T. Davis, 179 votes; B. F. Finkel, 257 votes; W. L. Hart, 310 votes; Miss Jewell C. Hughes, 149 votes; D. N. Lehmer, 293 votes; C. C. MacDuffee, 225 votes; E. R. Smith, 180 votes.

The following were accordingly declared elected: Vice-Presidents: E. T. BELL, California Institute of Technology; W. C. GRAUSTEIN, Harvard University.

Additional members of the Board of Trustees: A. A. BENNETT, Brown University; B. F. FINKEL, Drury College; W. L. HART, University of Minnesota; D. N. LEHMER, University of California.

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DEC. 16, 1929

RECEIPTS		EXPENDITURES		
Balance Dec. 17, 1928	$9,876.02	Publisher's bills (Nov. '28–Oct. '29)		$5,030.47
1928 indiv. dues	730.95	President's office		34.09
1928 instit. dues	23.10	Manager's office		31.42
1929 indiv. dues	6,826.45	Editor-in-Chief's office		453.49
1929 instit. dues	811.10	Committee on Geometry		124.95
1929 subscriptions	943.84	*Register*		115.00
Initiation fees	226.00	Secretary-Treasurer's office:		
Advertising	583.32	Postage	$309.56	
Sale copies of MONTHLY	46.15	Bond	5.00	
Sale First Carus Mon.	13.75	Safety deposit	4.00	
Sale Second Carus Mon.	17.50	Office supplies	125.17	
Sale Third Carus Mon.	28.75	Express, tel., etc.	58.59	
Sale Fourth Carus Mon.	51.25	Ins. back copies of MONTHLY	17.40	
Sale reprints	8.87	Clerical work	851.34	
For *Annals* subscription	3.50	Printing	279.80	
Sale Rhind Papyrus	151.00	Library expense	30.50	
Contributions A.B. Chace	4,000.00	Paid copies MONTHLY	97.40	
Refund New York meeting expense	25.25	New York Meeting	95.00	
Int. Oberlin Savgs. Bk.	157.56	Boulder Meeting	100.00	

Int. Peoples Bkg. Co.	115.74	Refund 1929 dues	7.00
Int. Liberty Bonds	85.00	Refund subscription	3.15
Int. Hardy Fund	120.00	Refund Papyrus	15.00
Int. certifs. of deposit	64.33	Forwarded for members' *Annals*	5.00
Int. from Genl. Endowment Fund Bonds	155.00	Paid members' *Annals*	12.00
Int. Chace Fund	50.00	Forwarded for reprints	8.87
Int. Chauvenet Fund	25.00		2,024.78
Int. from investment of current funds	150.00	*Annals* subvention	375.00
	15,413.41	Paid to sections from initiation fees	112.68
		Transferred to Genl. Endowment	750.00
Total 1929 receipts	$25,289.43	Cost above par of new bonds for Genl. Endowment Fund	122.09
		Pd. Drury College int. Hardy Fund, credit B. F. Finkel	120.00
		Sustaining memb. in Amer. Math. Soc.	100.00
		Printing Rhind Papyrus	4,000.00
		Expense acct. *Bibl. Math.*	118.75
		Expense acct. Carus Mons.	13.78
		Bought bond for Chace Fund	1,000.00
		Transfer to Chace Fund	416.00
		Transfer to Carus Mon. Fund	122.05
Total expenditures	$15,064.55	Total expenditures	15,064.55
		Cash on hand	13.28
Balance to the end of 1929 business	$10,224.88	Checking account	140.64
		Oberlin Savgs. Bk. acct.	2,845.77
		Peoples Bkg. Co. acct.	2,981.37
		Liberty Bonds	1,000.00
		5% Bonds, Harris Trust Co.	3,000.00
Received on 1930 business	996.50	Certif. of deposit	1,240.32
Book balance Dec. 16, 1929	$11,221.38	Bank balance Dec. 16, 1929	$11,221.38

EXHIBIT OF THE FUNDS OF THE ASSOCIATION

Carus Monograph Fund

Balance Dec. 17, 1928		$4,292.19
Receipts: Sales	$111.25	
Interest	171.23	282.48
		$4,574.67
Expenditures		13.78
		$4,560.89
Certificates of deposit	$4,510.89	
Cash in bank	50.00	
Balance Dec. 16, 1929		$4,560.89

ARNOLD BUFFUM CHACE FUND

Balance Dec. 17, 1928		$1,230.00
Receipts: From advance sales	$151.00	
Interest	50.00	201.00
		$1,431.00
Refund		15.00
		$1,416.00
Iowa Rwy & Light Co. 5% Bond	$1,000.00	
Certificate of deposit	416.00	
Balance Dec. 16, 1929		$1,416.00

CHAUVENET PRIZE FUND

Balance Dec. 17, 1928		$560.00
Interest	$25.00	
Amount set aside for 1929	20.00	45.00
		$605.00
Iowa Rwy & Light Co. 5% Bond	$500.00	
Cash in bank	105.00	
Balance Dec. 16, 1929		$605.00

LIFE MEMBERSHIP FUND

Liability on life memberships Dec. 17, 1928	$472.04
To be transferred to current funds, surplus	12.84
Liability on life memberships as of Jan. 1, 1930	$459.20

GENERAL ENDOWMENT FUND

Balance Dec. 17, 1928		$4,250.00
Transferred from current funds in 1929		750.00
		$5,000.00
Liberty Bond	$1,000.00	
Land Trust Certificate	1,000.00	
Cleveland Trust Securities Co. Gold Bond	1,000.00	
Idaho Power Co. 5% Bonds	2,000.00	
Balance Dec. 16, 1929		$5,000.00

Of the funds on hand, indicated in the first division of the financial report, $50.00 belongs to the Carus Monograph Fund (not yet transferred); $105.00 belongs to the Chauvenet Prize Fund; and $459.20 is held as a Life Membership Fund, representing the liability on life memberships already paid for, as of date Jan. 1, 1930. Aside from these amounts, the Carus Monograph Fund, to the amount of $4,510.89 is carried as a separate fund in the form of two certificates of deposit which bear 4%, compounded quarterly; $1,000.00 of the Arnold Buffum Chace Fund is carried in securities as listed above, and $416.00 more in the form of a certificate of deposit; the contribution of Professor W. B. Ford to the Chauvenet Fund is carried as a 5% Gold Bond; and the sum of $5,000.00

is held in reserve as the General Endowment Fund, in securities as listed above.

When the accounts were closed Dec. 16, 1929, there remained on the total business for the year 1929 the following items:

Bills Receivable (partly estimated)		Bills Payable (partly estimated)	
1929 individual dues	$150.00	Publisher's bills (Nov., Dec. 1929)	$1,150.00
Advertising	100.00	President's office	30.00
	$250.00	Manager's office	40.00
		Editor-in-chief's office	100.00
		Other editors' postage	50.00
		Register	500.00
		Secretary-Treasurer's office	300.00
		Annals subvention	150.00
		Initiation fees due to sections	750.00
		Carus Monograph Fund	50.00
		Chauvenet Prize Fund	105.00
		Life Membership Fund	459.20
		Typewriter	70.00
		Printing catalog of library	400.00
			$4,079.20

If to the balance on 1929 business shown in this report, $10,224.88, there be added the bills receivable, $250.00, and there be subtracted the estimated bills payable, $4,079.20, there results an estimated final balance on 1929 business of approximately $6,400, which represents the accumulated surplus in current funds, as compared with the corresponding figure of $5,190 of a year ago. This shows a healthy condition of the Association for which we may be gratified at a time when many banks and other organizations are in financial difficulties.

W. D. Cairns, *Secretary-Treasurer*

THE FOUNDATIONS OF THE IDEAL THEORY OF ZOLOTAREV[1]

By N. Tchebotarev

Let us recall some definitions and theorems on algebraic integers. Suppose that the algebraic number ω is a root of the irreducible equation

$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_n = 0 \tag{1}$$

with rational coefficients.

Definition 1: If the coefficients in (1) are rational integers then ω is called an algebraic integer.

The following theorems follow directly. Proofs may be found in any book on algebraic numbers.

[1] Translated from the Russian by S. Skolnik; edited and revised by H. T. Engstrom. This version is not a literal translation, but has been adapted to English-speaking readers. E. T. Bell.

I. Every algebraic integer which is rational is a rational integer.

II. Every algebraic number satisfying an equation with integral rational coefficients and leading coefficient unity is an algebraic integer. (Theorem of Gauss).

III. Every algebraic number satisfying an equation with integral algebraic coefficients and leading coefficient unity is an algebraic integer.

IV. The integers of an algebraic field form a ring.

Definition 2: The product of the conjugates of an algebraic number ω is called its norm, $N(\omega)$; i.e.

$$N(\omega) = \omega \cdot \omega^{(1)} \cdots \omega^{(n-1)}. \tag{2}$$

It follows immediately that

$$N(\omega_1 \cdot \omega_2) = N(\omega_1)N(\omega_2). \tag{3}$$

Furthermore if ω satisfies (1) and t is a rational number

$$N(t - \omega) = \pm f(t). \tag{4}$$

Definition 3: A system of n integers $\omega_0, \omega_1, \cdots, \omega_{n-1}$ of an algebraic field K of n-th degree is called an integral base of K if every integer of the field can be represented in the form

$$c_0\omega_0 + c_1\omega_1 + \cdots + c_{n-1}\omega_{n-1} \tag{5}$$

with rational integral coefficients $c_0, c_1, \cdots, c_{n-1}$.

V. There exists an integral base in every algebraic field.

2.

The theory of algebraic integers cannot be immediately developed as in rational number theory since the uniqueness of prime decomposition is not valid. To remedy this difficulty we introduce the notion of divisibility for a rational prime modulus p.

Definition 4: One algebraic integer, ω_1, is divisible by another, ω_2, for the modulus p if there exists a rational integer c, prime to p, such that $c\omega_1/\omega_2$ is an algebraic integer.

If $N(\omega)$ is prime to p, then all integers of the field are divisible (mod p) by ω. For, since $N(\omega)\,\omega_1/\omega$ is an integer, we may choose $c = N(\omega)$.

If ω_1 is divisible (mod p) by ω_2, the power of p contained in $N(\omega_1)$ is not less than the power contained in $N(\omega_2)$. For there exists a c such that $(c, p) = 1$ and $c\omega_1/\omega_2$ is an integer. Hence its norm $c^n N(\omega_1)/N(\omega_2)$ is an integer, and the theorem follows.

If ω_1 and ω_2 are each divisible by the other (mod p) we shall call them associate (mod p) and we shall regard them as identical in questions of divisibility (mod p).

Definition 5: The algebraic integer π is a prime (mod p) if π can divide a product $\omega_1\omega_2$ only when it divides at least one of the factors ω_1, ω_2.

We shall determine the decomposition of p in the field in the form

$$c \cdot p = \pi_1 \cdot \pi_2 \cdots \pi_k, \tag{6}$$

where $(c, p) = 1$ and $\pi_i (i = 1, 2, \cdots, k)$ is prime (mod p). Let us consider the expression

$$c_0\omega_0 + c_1\omega_1 + \cdots + c_{n-1}\omega_{n-1}, \tag{7}$$

where $\omega_0, \omega_1, \cdots, \omega_{n-1}$ is an integral base of the field. By giving to each of the coordinates $c_0, c_1, \cdots, c_{n-1}$ the values $0, 1, 2, \cdots, p-1$ (mod p) we obtain $\sigma = p^n$ different numbers $\alpha_0 \equiv 0, \alpha_1, \alpha_2, \cdots, \alpha_{\sigma-1}$.

If $N(\alpha_i)$ is prime to p for all $i \neq 0$ then p is a prime in the field. To prove this let us note that every integer of the field can be written in the form $\alpha_i + p\omega'$, where ω' is an integer. Hence

$$N(\omega) = N(\alpha_i) + p\Omega, \tag{8}$$

where Ω is an integer, i.e.,

$$N(\omega) \equiv N(\alpha_i) \pmod{p}. \tag{9}$$

Therefore $N(\omega)$ is divisible by p only if $N(\alpha_i)$ is divisible by p, i.e., according to our hypothesis, if $\alpha_i \equiv 0$. That is, $N(\omega)$ is divisible by p only if ω is divisible by p. Suppose now that $\omega_1\omega_2$ is divisible (mod p) by p, i.e. $c\,\omega_1\omega_2/p$ is an integer where $(c, p) = 1$. Then its norm $c^n N(\omega_1)N(\omega_2)/p^n$ is a rational integer and hence either $N(\omega_1)$ or $N(\omega_2)$ must be divisible by p, i.e., either ω_1 or ω_2 must be divisible by p. Hence p is a prime in the field.

As a preliminary to an important theorem of Zolotarev, let us choose an arbitrary integer ω and consider the integers $\omega + p^r\omega'$ where ω' runs through the integers of the field. Among these integers we choose one, $\Omega = \omega + p^r\Omega'$, whose norm contains the least power of p. This choice can be made from the finite set for which

$$\omega' = c_0'\omega_0 + c_1'\omega_1 + \cdots + c_{n-1}'\omega_{n-1}, \tag{10}$$

where each of the coordinates $c'_0, c'_1, \cdots, c'_{n-1}$ takes the values $0, 1, 2, \cdots p^{\lambda-r+1} - 1$, and p^λ is the power of p contained in $N(\omega)$. For Ω' may be written in the form

$$\Omega' = \omega' + p^{\lambda-r+1}\omega'', \tag{11}$$

where ω' is one of the numbers (10), i.e.,

$$\Omega = (\omega + p^r\omega') + p^{\lambda+1}\omega''.$$

Hence

$$N(\Omega) \equiv N(\omega + p^r\omega') \pmod{p^{\lambda+1}}. \tag{12}$$

It follows that $N(\Omega)$ and $N(\omega + p^r\omega')$ contain the same power of p. In other words, we may assume $\Omega = \omega + p^r\omega'$ where ω' is one of the set (10).

The generalized theorem of Zolotarev is the following:

Theorem: If among the norms $N(\omega+p^{\nu}\omega')$, where ω' runs over all integers of the field, $N(\omega)$ contains the least power of p then p^{ν} is divisible (mod p) by ω.

Suppose ω satisfies the conditions of the theorem and is a root of the equation

$$(13)\quad f(\omega) = \omega^n + p^{\lambda_{n-1}}\cdot c_1\omega^{n-1} + p^{\lambda_{n-2}}\cdot c_2\omega^{n-2} + \cdots + p^{\lambda_1}c_{n-1}\omega + p^{\lambda}c_n = 0,$$

where $c_1, c_2, \cdots, c_n$ are prime to p. Let $\mu = r/s$ be the greatest of the numbers

$$\frac{\lambda-\lambda_1}{1}, \frac{\lambda-\lambda_2}{2}, \cdots, \frac{\lambda-\lambda_{n-1}}{n-1}, \frac{\lambda-\lambda_n}{n},$$

and suppose $(r, s) = 1$.

We shall prove that $p^{\mu}c_n/\omega$ is an integer. Multiplying (13) by $p^{n\mu-\lambda}\cdot c_n^{n-1}/\omega^n$ and rewriting in the reverse order, we see that $p^{\mu}c_n/\omega$ satisfies the equation

$$(14)\quad \left(\frac{c_np^{\mu}}{\omega}\right)^n + p^{\lambda_1+\mu-\lambda}c_{n-1}\left(\frac{c_np^{\mu}}{\omega}\right)^{n-1} + \cdots + p^{\lambda_{n-1}+\mu(n-1)-\lambda}c_1\cdot c_n^{n-1}\left(\frac{c_np^{\mu}}{\omega}\right) + p^{\mu n-\lambda}c_n^{n-1} = 0,$$

where the coefficients are algebraic integers since, by definition, $\mu \geqq (\lambda-\lambda_h)/h$, or $\lambda_h+\mu h-\lambda \geqq 0$. [The coefficients in (14) may not be rational since μ may be fractional.] It follows that

$$(c_np^{\mu}/\omega)^s = c_n^{s}p^{r}/\omega^s$$

is also an integer of the field.

Since $N(\omega)$ contains p^{λ}, the norms of all numbers

$$(15)\quad \zeta = \omega - p^{\nu}\left(\frac{c_n^{s}p^{r}}{\omega^s}\right)^{i} = \omega - \frac{p^{ri+\nu}c_n^{si}}{\omega^{si}} \qquad (i = 1, 2, \cdots)$$

will contain p to a degree not less than λ. But

$$N(\zeta) = N\left(\omega - \frac{p^{ri+\nu}c_n^{si}}{\omega^{si}}\right) = \frac{N(\omega^{si+1} - p^{ri+\nu}c_n^{si})}{[N(\omega)]^{si}}.$$

Hence it follows that $N(\omega^{si+1} - p^{ri+\nu}c_n^{si})$ is divisible by p to a degree at least equal to $\lambda(s_i+1)$.

If $\epsilon = e^{2\pi i/(si+1)}$ is a primitive root of unity, we have the identity

$$x^{si+1} - y^{si+1} = (x-y)(x-\epsilon y)\cdots(x-\epsilon^{si}y).$$

By applying this identity to $\omega^{si+1} - p^{ri+\nu}\cdot c_n^{si}$ and taking the norm we obtain

$$(16)\quad N(\omega^{si+1} - p^{ri+\nu}c_n^{si}) = \prod_{j=0}^{si} N(\omega - \epsilon^j p^{(ri+\nu)/(si+1)}c_n^{si/(si+1)}),$$

where in taking the norms, the quantities ϵ, $p^{(ri+\nu)/(si+1)}$, $c_n^{si/(si+1)}$ must be treated as rational numbers. Applying formula (4), we have

$$N(\omega - \epsilon^i p^{(ri+\nu)/(si+1)} \cdot c_n^{si/(si+1)}) = \pm f(\epsilon^i \cdot p^{(ri+\nu)/(si+1)} c_n^{si/(si+1)}). \tag{17}$$

Performing the substitutions on the right, the successive terms from (13) will contain p with degrees

$$n \cdot \frac{ri+\nu}{si+1}, \quad \lambda_{n-1} + (n-1)\frac{ri+\nu}{si+1}, \cdots, \lambda_1 + \frac{ri+\nu}{si+1}, \lambda, \tag{18}$$

respectively. If we choose i in (15) so that no two of (18) are equal, then the degree of p contained in the norm (17) will be the least of the numbers (18). We may choose i so that no two are equal except in case the equality

$$h\frac{ri+\nu}{si+1} + \lambda_h = k\frac{ri+\nu}{si+1} + \lambda_k, h \neq k,$$

is satisfied for every i. That is

$$(h-k)(ri+\nu) = (\lambda_k - \lambda_h)(si+1),$$

or

$$(h-k)r = (\lambda_k - \lambda_h)s \quad \text{and} \quad (h-k)\nu = (\lambda_k - \lambda_h).$$

Hence i may be properly chosen unless $\nu = (r/s) = \mu$. We shall exclude this case for the present.

We are now able to show that $\mu \leqq \nu$. For suppose $\mu > \nu$, i.e., $r > s\nu$. Adding rsi to both sides of the inequality we have

$$r(si+1) > s(ri+\nu) \quad \text{or} \quad \mu = \frac{r}{s} > \frac{ri+\nu}{si+1}. \tag{19}$$

On the other hand, from the definition of μ, there exists a subscript f for which

$$\mu = (\lambda - \lambda_f)/f \quad \text{or} \quad \lambda = \lambda_f + \mu f.$$

Hence, from (19),

$$\lambda > \lambda_f + f(ri+\nu)/(si+1). \tag{20}$$

Since the expression on the right is one of (18) we conclude that (17), for each j, contains p to a degree less than λ. That is, the product (16) contains p to a degree less than $\lambda(si+1)$. This contradicts the statement made above. Hence $\mu \leqq \nu$. The excluded case, $\mu = \nu$, is included in the conclusion.

It follows immediately that $(p^\nu \cdot c_n/\omega) = p^{\nu-\mu} \cdot (p^\mu \cdot c_n/\omega)$ is an algebraic integer, i.e., p^ν is divisible by ω (mod p). Q. E. D.

Applying this theorem to the case $\nu = 1$, it is seen that $\alpha_1, \alpha_2, \cdots, \alpha_{s-1}$ may be chosen so that p is divisible (mod p) by each of them.

We shall now prove the existence of the greatest common divisor (mod p) of two integers ω_1 and ω_2. Let ω_3 be an integer of the form $\omega_1 + \omega_2\omega'$ whose norm contains the least power of p. As above, it is sufficient to consider ω' of the

form $c_0'\omega_0+c_1'\omega_1+\cdots+c_{n-1}'\omega_{n-1}$ where each c' takes on the values 0, 1, 2, $\cdots$, p^r-1 and p^r is the highest power of p dividing $N(\omega_1)$ or $N(\omega_2)$.

Let p^r be the least power of p divisible (mod p) by ω_2. It exists, since if $N(\omega)=c\cdot p^{r'}$ then $p^{r'}$ is divisible (mod p) by ω, and it may be determined in a finite number of steps. By definition of ω_3, among the norms $N(\omega_3+\omega_2\omega')$ the norm $N(\omega_3)$ contains the least power of p. Multiplying by the integer $\omega_4=(cp^r/\omega_2)$, it follows that of all norms of the type

$$N[\omega_4(\omega_3+\omega_2\omega')] = N(\omega_4\omega_3+\rho^r c\omega')$$

the norm of $\omega_4\omega_3$ contains the least power of p. But ω' is equivalent to $c'\omega''$, where ω'' runs through all integers of the field and

$$c\cdot c' \equiv 1 \pmod{p^\mu}$$

for sufficiently large μ. Hence, of all norms $N(\omega_4\cdot\omega_3+p^r\cdot c\cdot\omega')$, the norm of $\omega_4\omega_3$ contains the least power of p, and, by the generalized theorem of Zolotarev, it follows that p^r is divisible (mod p) by $\omega_4\cdot\omega_3$. But $cp^r=\omega_2\cdot\omega_4$, and hence, for some c'',

$$c''\cdot c\cdot\omega_2\cdot\omega_4/(\omega_4\cdot\omega_3) = c''\cdot c\cdot\omega_2/\omega_3$$

is an integer i.e., ω_2 is divisible (mod p) by ω_3. It follows directly that $\omega_1=\omega_3-\omega_2\cdot\omega'$ is divisible (mod p) by w_3.

The form of $\omega_3=\omega_1+\omega_2\omega'$ indicates that every divisor of ω_1 and ω_2 is also a divisor of ω_3, and therefore the term "greatest common divisor" is justified. It is readily proved that an interchange of ω_1 and ω_2 will lead to an associate of ω_3.

If ω' can be chosen so that $N(\omega_1+\omega_2\omega')$ is not divisible by p, then ω_1 and ω_2 will be called relatively prime (mod p).

We shall prove the following lemmas concerning divisibility (mod p).

Lemma 1: If $\alpha\beta$ is divisible by γ, and α is prime to γ, then β is divisible by γ.

In fact, if α and β are relatively prime there exists an integer $\epsilon=\alpha+\gamma\omega'$ such that $N(\epsilon)$ is not divisible by p. Multiplying by the integer $\epsilon'\beta=(c/\epsilon)\beta$, we obtain

$$\beta\cdot c = \alpha\beta\cdot\epsilon' + \gamma\cdot\epsilon'\beta\cdot\omega'.$$

But, by hypothesis, $c'\alpha\beta/\gamma$ is an integer. Hence

$$(\beta/\gamma)c\cdot c' = (\alpha\beta/\gamma)\epsilon\cdot c' + \epsilon'\beta\cdot\omega' c'$$

is an integer, i.e., β is divisible (mod p) by γ.

Lemma 2. If α and β are both prime to γ then $\alpha\beta$ is prime to γ.

For we may choose $\epsilon'=\alpha+\gamma\omega'$ and $\epsilon''=\beta+\gamma\omega''$ so that $N(\epsilon)$ and $N(\epsilon')$ are not divisible by p. Multiplying, we obtain

$$\epsilon\cdot\epsilon' = \alpha\beta + \gamma(\alpha\omega'' + \beta\omega' + \gamma\omega'\cdot\omega'')$$

where $N(\epsilon\cdot\epsilon')$ is not divisible by p, i.e., $\alpha\beta$ is prime to γ.

Lemma 3: If α is divisible by two relatively prime integers β and γ then it is divisible by $\beta \cdot \gamma$.

For, if α is divisible by β, then, for some c prime to p, $c\alpha/\beta$ is an integer. But $c\alpha = (c\alpha/\beta)\beta$ is divisible by γ. Hence, since β is prime to γ, it follows that $c\alpha/\beta$ is divisible by γ. That is, for some c' prime to p, $c'c\alpha/(\beta\gamma)$ is an integer. Hence α is divisible by $\beta\gamma$.

If π is a prime (mod p) then every integer of the field is either divisible by π or prime to π. For if ω is not divisible by π, by Definition 5, it follows that π and ω have no common divisor (mod p).

We are now prepared to factor p into its prime factors (mod p). The factors may be found in the system $\alpha_1, \alpha_2, \cdots, \alpha_{\sigma-1}$. For every integer of the field can be written in the form $\alpha_i + p\omega'$ where p is divisible by α_i and hence $\alpha_i + p\omega'$ is divisible by α_i (mod p).

If the norms of $\alpha_1, \alpha_2, \cdots, \alpha_{\sigma-1}$ are prime to p, it follows directly that p is a prime (mod p) in the field.

Suppose that some of the norms $N(\alpha_1), N(\alpha_2), \cdots, N(\alpha_{\sigma-1})$ are divisible by p. Then if π_1 is that α whose norm contains the least power of p, say p'^1, then π_1 is a prime. To prove this, let us show that any integer ω of the field is either divisible by π_1 or prime to it. If there exists a number of the type $\pi_1 + \omega \cdot \omega'$ whose norm does not contain p, then π_1 is prime to ω. On the other hand if $N(\pi_1 + \omega \cdot \omega')$, for arbitrary ω', is divisible by p it must be divisible by p'^1, since any such integer can be expressed as $\alpha_i + p\omega''$. Hence among all $N(\pi_1 + \omega \cdot \omega')$ the norm $N(\pi_1)$ contains the least power of p, i.e., ω is divisible (mod p) by π_1.

Let $\pi_1^{e_1}$ be the greatest power of π_1 dividing p (mod p). The integer $c \cdot p/\pi_1^{e_1}$ will not be divisible by π_1 and consequently will be prime to it.

Let us now choose among those of $\alpha_1, \alpha_2, \cdots, \alpha_{\sigma-1}$ which are prime to π_1 an integer π_2 whose norm contains the least power of p, say p'^2. Then π_2 is a prime (mod p). For let ω be an arbitrary integer of the field and consider all integers $\pi_2 + \pi_1\omega \cdot \omega'$. If there exists an integer $\pi_2 + \pi_1\omega \cdot \omega'$ whose norm does not contain p then π_2 is prime to ω. If not, $N(\pi_2 + \pi_1\omega\omega')$ will be divisible by at least p'^2 since we may write

$$\pi_2 + \pi_1\omega \cdot \omega' = \alpha_i + p\omega'',$$

where $\pi_1\omega\omega'$ and $p\omega''$ are divisible by π_1, and π_2 is prime to π_1, i.e., where α_i is prime to π_1. Hence $\pi_1\omega$ is divisible by π_2 and, by Lemma 1, π_2 divides ω.

If $\pi_2^{e_2}$ is the highest power of π_2 dividing p (mod p) then, by Lemma 3, p must be divisible by $\pi_1^{e_1} \cdot \pi_2^{e_2}$ and the quotient $cp/(\pi_1^{e_1} \cdot \pi_2^{e_2})$ will be prime to both π_1 and π_2.

By selecting from among those of $\alpha_1, \alpha_2, \cdots, \alpha_{\sigma-1}$ which are prime to $\pi_1 \cdot \pi_2$ the integer π_3 whose norm contains the least power of p we obtain another prime factor of p (mod p). Since each π_i is a distinct member of the set $\alpha_1, \alpha_2, \cdots, \alpha_{\sigma-1}$, the complete decomposition may be found in a finite number of steps. We have then

$$cp = \epsilon \cdot \pi_1^{e_1} \cdot \pi_2^{e_2} \cdot \cdots \pi_k^{e_k} \tag{21}$$

where c is prime to p. It is readily shown that this decomposition is unique in the sense that if we also have

$$c'p=\epsilon'\cdot\pi'_1{}^{e'_1}\pi_2'^{e'_2}\cdot\cdots\pi'_{k'}{}^{e'_{k'}} \tag{22}$$

then the factors of (21) and (22) must be associate in pairs. For we have

$$c'\epsilon\cdot\pi_1^{e_1}\cdot\pi_2^{e_2}\cdots\pi_k^{e_k}=c\epsilon'\cdot\pi_1'^{e'_1}\cdot\pi_2'^{e'_2}\cdots\pi'_{k'}{}^{e'_{k'}}. \tag{23}$$

Hence the left of (23) is divisible (mod p) by π_1' and one of the factors, say π_1, must be divisible by π_1'. But since both π_1 and π_1' are prime, each is divisible by the other and hence they are associate. Dividing (23) by π_1' and repeating the process it follows that $e_1=e_1'$. A similar conclusion follows for the other factors.

We shall now prove that $\pi_1, \pi_2, \cdots, \pi_k$ are the only primes (mod p) in the field. Let ω be an arbitrary integer of the field divisible (mod p) by $\pi_1^{m_1}, \pi_2^{m_2}, \cdots, \pi_k^{m_k}$ and therefore by their product. The quotient

$$\bar{\omega}=\frac{c'\cdot\omega}{\pi_1^{m_1}\cdot\pi_2^{m_2}\cdots\pi_k^{m_k}}$$

is prime with each π_i and consequently with $c\cdot p=\pi_1^{e_1}\cdot\pi_2^{e_2}\cdots\pi_k^{e_k}$, i.e., it is possible to choose an ω' so that $N(\bar{\omega}+p\omega')$ is not divisible by p. By (9), it follows that $\bar{c}=N(\bar{\omega})$ is prime to p. Multiplying $\bar{\omega}$ by $\overline{\bar{\omega}}=(\bar{c}/\bar{\omega})$ we have

$$\bar{c}=\frac{c'\cdot\omega\cdot\overline{\bar{\omega}}}{\pi_1^{m_1}\cdot\pi_2^{m_2}\cdots\pi_k^{m_k}},$$

i.e.,

$$\frac{\bar{c}\cdot\pi_1^{m_1}\cdot\pi_2^{m_2}\cdots\pi_k^{m_k}}{\omega}=c\cdot\overline{\bar{\omega}}.$$

Hence the product $\pi_1^{m_1}\cdot\pi_2^{m_2}\cdots\pi_k^{m_k}$ is divisible (mod p) by ω. Hence $\pi_1^{m_1}\cdot\pi_2^{m_2}\cdots\pi_k^{m_k}$ is the decomposition of ω into prime (mod p) factors.

Taking the norm of (21), we have

$$c^n\cdot p^n=[N(\pi_1)]^{e_1}[N(\pi_2)]^{e_2}[N(\pi_3)]^{e_3}\cdots[N(\pi_k)]^{e_k}. \tag{24}$$

It follows immediately from (24) that

$$n=e_1f_1+e_2f_2+\cdots+e_kf_k. \tag{25}$$

The number f_i is called the degree of π_i.

In the case where the discriminant of the equation $f(x)=0$ which defines the field is not divisible by p the decomposition of p may be found very simply. Let

$$f(x)\equiv f_1(x)f_2(x)f_3(x)\cdots f_k(x) \pmod{p}, \tag{26}$$

where $f_i(x)$, $i=1, 2, \cdots, k$, is irreducible (mod p). We may write

$$f(x) = f_1(x)f_2(x)\cdots f_k(x) + p\cdot\phi(x), \tag{27}$$

where $\phi(x)$ is not divisible by $f_i(x)$. Hence, if ω is a root of $f(x)=0$,

$$-p\phi(\omega) = f_1(\omega)f_2(\omega)\cdots f_k(\omega). \tag{28}$$

The numbers $\pi_i=f_i(\omega)$, $i=1, 2, \cdots, k$, are primes (mod p). For every integer α of the field can be written in the form

$$c\alpha = a_0 + a_1\omega + \cdots + a_{n-1}\omega^{n-1} = \psi(\omega),$$

where a and c are rational integers and $(c, p)=1$. The polynomial $\psi(x)$ is either divisible by $f_i(x)$ (mod p) or prime to it. In the first case α is divisible (mod p) by π_i, in the second case it is prime to π_i.

Let us note that the degree of the prime π_i will be equal to the degree of the corresponding polynomial. The method is also valid for some primes which divide the discriminant. The prime must not divide the index of ω, (i.e., it must not be an "ausserwesentlicher Diskriminantenteiler").

3.

Let us consider the problem of determining the divisors of an integer ω of the field for all rational prime moduli. We need consider as moduli only the rational primes dividing $N(\omega)$. (For other moduli the integer ω obviously plays the role of unit.)

Definition: To each prime divisor π (mod p) we associate a symbol $\mathfrak{P}$, called a prime divisor, and say that ω contains the divisor $\mathfrak{P}^k$ when ω is divisible (mod p) by π^k. The norm of $\mathfrak{P}$ is defined as the degree of p dividing $N(\pi)$. The product of several prime divisors we call a divisor.

Let us represent ω symbolically as the product of its divisors $\mathfrak{P}$ for all rational prime divisors of $N(\omega)$ as moduli, and define the norm of a product of prime divisors as the product of the norms. We see that $N(\omega)$ is equal to the product of the norms of the divisors of ω.

We shall make use of the following lemmas:

Lemma 1: If α is divisible (mod p) by β we may determine an integer $c\alpha/\beta$ where $(c, p)=1$ and c divides $N(\beta)$.

For let $\lambda=(c\alpha/\beta)$ be an integer. Then $\mu=(N(\beta)\alpha/\beta)$ is also an integer. The equation $c\cdot X - N(\beta)\cdot Y = d$, where d is the greatest common divisor of c and $N(\beta)$ has an integral rational solution X, Y. Then $\lambda X - \mu Y = (d\alpha/\beta)$ is an integer and $(d, p)=1$ since d divides c. Furthermore d divides $N(\beta)$.

Lemma 2: If α is divisible by β for all rational prime divisors of $N(\beta)$ as moduli, then α is divisible by β algebraically.

Let $N(\beta)=p_1^{w_1}p_2^{w_2}\cdots p_k^{w_k}$. By Lemma 1 there exist integers of the field

$$\lambda_1 = \frac{M_1\alpha}{\beta}, \lambda_2 = \frac{M_2\alpha}{\beta}, \cdots \lambda_k = \frac{M_k\alpha}{\beta},$$

where M_i, divides $N(\beta)/p_i^{w_i}$, $i=1, 2, \cdots, k$. Hence $M_1, M_2, \cdots, M_k$ have no common divisor and

$$M_1X_1 + M_2X_2 + \cdots + M_kX_k = 1$$

has a solution in rational integers. Then

$$\lambda_1X_1 + \lambda_2X_2 + \cdots + \lambda_kX_k = \frac{\alpha}{\beta}$$

is an integer, i.e., α is algebraically divisible by β.

Lemma 3: If α and β are relatively prime for all rational primes as moduli which divide both $N(\alpha)$ and $N(\beta)$, then they are relatively prime algebraically i.e., there exist integers λ and μ such that

$$\alpha\lambda + \beta\mu = 1. \tag{29}$$

For let $p_1, p_2, \cdots, p_k$ be the primes dividing both $N(\alpha)$ and $N(\beta)$. By hypothesis there exists an integer ω_i for each i such that

$$\alpha + \beta\omega_i = \epsilon_i,$$

where $N(\epsilon_i)$ is prime to p_i. Multiplying by $\epsilon_i' = (N(\epsilon_i)/\epsilon_i)$ we have

$$\alpha\rho_i + \beta\sigma_i = N(\epsilon_i), \tag{30}$$

where ρ_i and σ_i are integers of the field. We also have

$$\begin{aligned} \alpha\cdot\alpha' &= N(\alpha), \\ \beta\cdot\beta' &= N(\beta), \end{aligned} \tag{31}$$

where α' and β' are integers. Since $N(\epsilon_1), N(\epsilon_2), \cdots, N(\epsilon_k), N(\alpha), N(\beta)$ have no common divisor, the equation

$$N(\epsilon_1)X_1 + N(\epsilon_2)X_2 + \cdots + N(\epsilon_k)X_k + N(\alpha)Y_1 + N(\beta)Y_2 = 1$$

has an integral rational solution. Hence, from (30) and (31)

$$\begin{aligned} \alpha(\rho_1X_1 + \rho_2X_2 + \cdots + \rho_kX_k + \alpha'Y_1) \\ + \beta(\sigma_1X_1 + \sigma_2X_2 + \cdots + \sigma_kX_k + \beta'Y_2) = 1, \end{aligned} \tag{32}$$

i.e., we have an equality of type (29).

Let us now prove the theorem:

Theorem: The algebraic integer ω is defined by its divisors to within an algebraic unit.

For if two integers ω and ω' have the same divisors each must be divisible by the other for all rational prime moduli which divide $N(\omega) = \pm N(\omega')$ and hence, by Lemma 2, each must divide the other algebraically, i.e., their quotient is an algebraic unit.

The following theorem establishes the independence of the divisors.

Theorem: Given a divisor $\mathfrak{M}$ and a rational integer M, an algebraic integer μ may be determined so that $\mu/\mathfrak{M}$ is prime to M, (i.e., its norm, $N(\mu)/N(\mathfrak{M})$, is prime to M.)

Let us first prove the theorem for a prime divisor $\mathfrak{P}$ dividing $\mathfrak{M}$. Let π be the prime (mod p) corresponding to $\mathfrak{P}$ and let $N(\mathfrak{P})=p^k$, $N(\pi)=c\cdot p^k$ where $(c,\ p)=1$. Suppose $M=M'p^m$ where $(M',\ p)=1$. Then $\pi'=M'\pi+p^{k+1}$ will satisfy the condition of the theorem since

$$N(\pi') \equiv N(p^{k+1}) = p^{n(k+1)} \pmod{M'},$$

$$N(\pi') \equiv M'^n \cdot c \cdot p^k \pmod{p^{k+1}}.$$

To determine a μ for the divisor $\mathfrak{M}$, construct such integers for each of its prime divisors and multiply them.

Let us compare the divisor concept with Dedekind's ideal numbers. Dedekind's definition of an ideal is the following:

A set of integers of a field is called an ideal if: (a) The sum of any two integers of the set belongs to the set. (b) The product of any integer of the set with an arbitrary integer of the field belongs to the set.

We shall prove that the concept of an ideal is equivalent to the concept of the set of all integers of the field divisible by a given divisor. It is obvious that the set of all integers divisible by a given divisor is an ideal. Let us prove the converse. We choose an ideal $\overline{\mathfrak{M}}$ by means of its base $\mu_1, \mu_2, \cdots, \mu_n$ and determine the common divisor $\mathfrak{M}$ of $\mu_1, \mu_2, \cdots, \mu_n$. It will be necessary to consider the rational primes $p_1, p_2, \cdots$ which divide the greatest common divisor D of $N(\mu_1), N(\mu_2), \cdots, N(\mu_n)$. Let

$$\mathfrak{M} = (\mathfrak{P}_{11}^{w_{11}} \cdot \mathfrak{P}_{12}^{w_{12}} \cdots \mathfrak{P}_{1k_1}^{w_{1k_1}})(\mathfrak{P}_{21}^{w_{21}} \mathfrak{P}_{22}^{w_{22}} \cdots \mathfrak{P}_{2k_2}^{w_{2k_2}}) \cdots \tag{33}$$

where $\mathfrak{P}_{ji}$ corresponds to prime divisors for the modulus p_j. Clearly each number in $\overline{\mathfrak{M}}$ will be divisible by the divisor $\mathfrak{M}$. We must show that if an integer ω of the field is divisible by $\mathfrak{M}$, then ω belongs to $\overline{\mathfrak{M}}$. Let us consider separately each rational prime modulus in $N(\mathfrak{M})$. Suppose $\lambda=b_1\mu_1+b_2\mu_2+\cdots+b_n\mu_n$ is an integer in $\overline{\mathfrak{M}}$ divisible by $\mathfrak{P}_{11}^{w_{11}} \mathfrak{P}_{12}^{w_{12}} \cdots \mathfrak{P}_{1k_1}^{w_{1k_1}}$ and having no other divisors (mod p_1). Such an integer must exist by definition of $\mathfrak{M}$. Then ω is divisible (mod p_1) by λ, i.e.,

$$c_1\omega = c_{11}\mu_1 + c_{12}\mu_2 + \cdots + c_{1n}\mu_n. \tag{34}$$

where $(c_1,\ p_1)=1$. Similarly

$$\begin{aligned} c_2\omega &= c_{21}\mu_1 + c_{22}\mu_2 + \cdots + c_{2n}\mu_n, \\ c_3\omega &= c_{31}\mu_1 + c_{32}\mu_2 + \cdots + c_{3n}\mu_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{34}$$

where $(c_i,\ p_i)=1$. The integers $N(\mu_1), N(\mu_2), \cdots, N(\mu_n)$ and hence their greatest common divisor D belong to the ideal $\mathfrak{M}$. But D contains only the

prime divisors of $N(\mathfrak{M})$. Hence D, c_1, c_2, $\cdots$, have no common divisor and the equation

$$DY + c_1X_1 + c_2X_2 + \cdots = 1 \tag{35}$$

has an integral rational solution. But $D\omega$ belongs to $\overline{\mathfrak{M}}$ with D, i.e.,

$$D\omega = d_1\mu_1 + d_2\mu_2 + \cdots + d_n\mu_n. \tag{36}$$

Multiplying (36) by Y and (34) by X_1, X_2, $\cdots$, respectively, we obtain

$$\omega = (d_1Y + c_{11}X_1 + c_{21}X_2 + \cdots)\mu_1 + (d_2Y + c_{12}X_1 + c_{22}X_2 + \cdots)\mu_2 + \cdots + (d_nY + c_{1n}X_1 + c_{2n}X_2 + \cdots)\mu_n.$$

Hence ω belongs to $\overline{\mathfrak{M}}$. Q. E. D.

The theory of congruences for an ideal modulus may be carried through very simply by this method.

AN EXAMPLE OF THE IDEAL THEORY OF ZOLOTAREV

By H. T. ENGSTROM[1]

In the following note the prime ideal decomposition of a common index divisor (gemeinsamer ausserwesentlicher Diskriminantenteiler) is determined to illustrate the method of Zolotarev. The index divisors are the exception in Dedekind's theorem for the determination of the prime ideal decomposition of a rational prime p from the decomposition (mod p) of an equation defining the field.

Let us consider the field K generated by a root θ of the irreducible equation $f(x)=x^3-x^2-2x-8=0$. This field is given by Bachmann[2] as an example of a field having a common index divisor. The discriminant d_θ of θ is $-2^2\cdot 503$, and since $f(x)=x^2(x-1)+2(x-4)$, where $x-4\equiv 0 \pmod{2, x}$ it follows that 2 divides the index of θ. We shall determine the ideal decomposition of 2.

The integers $1, \theta, \eta=\frac{1}{2}\theta(\theta-1)-1$ form a base of K. Furthermore if $\alpha=a+b\theta+c\eta$, then

$$N(\alpha) = \begin{vmatrix} a & b & c \\ 2b+4c & a+b & 2b \\ 4b-2c & 2c & a-c \end{vmatrix}.$$

From $N(\alpha)$ we determine an integer in each congruence class (mod 2) whose norm contains the least power of 2 and obtain the set,

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[1] National Research Fellow, California Institute of Technology.

[2] P. Bachmann, *Allgemeine Arithmetik der Zahlenkörper*, 1926, pp. 280–284.

$$\alpha_0 = 0,\ \alpha_1 = 1,\ \alpha_2 = \theta + 2\eta,\ \alpha_3 = 2\theta + \eta,\ \alpha_4 = -1 + \theta,$$
$$\alpha_5 = 1 + \eta,\ \alpha_6 = 2 + \theta + \eta,\ \alpha_7 = 1 - \theta + \eta,$$

where

$$N(\alpha_0) = 0,\ N(\alpha_1) = 1,\ N(\alpha_2) = 100,\ N(\alpha_3) = 68,$$
$$N(\alpha_4) = 10,\ N(\alpha_5) = 10,\ N(\alpha_6) = 2,\ N(\alpha_7) = -4.$$

Since $N(\alpha_4)$ contains exactly 2^1 it follows from the general theory that 2 is divisible (mod 2) by the prime divisor $\alpha_4 = -1+\theta$. Furthermore $-1+\theta$ and $1+\eta$ are relatively prime (mod 2) since by choosing $a=0$, $b=1$, $c=3$ in the expression

$$\beta = 1 + \eta + (-1 + \theta)(a + b\theta + c\eta)$$

we obtain $\beta = 11$; i.e., the greatest common divisor (mod 2) of $-1+\theta$ and $1+\eta$ is a unit. Hence 2 is also divisible by the prime $1+\eta$. Similarly it follows that $2+\theta+\eta$ is prime (mod 2) to both $-1+\theta$ and $1+\eta$ and divides 2. Since $N(2) = 2^3$, there must exist a rational integer d prime to 2 and a unit ϵ (mod 2) of K such that

$$d \cdot 2 = \epsilon(2 + \theta + \eta)(-1 + \theta)(1 + \eta).$$

By writing $\epsilon = a+b\theta+c\eta$ and multiplying out it is seen that we may choose $\epsilon = 5+2\theta-6\eta$ and $d=25$. Hence in the prime ideal divisors of Zolotarev,

$$2 = \mathfrak{P}_1 \cdot \mathfrak{P}_2 \cdot \mathfrak{P}_3,\ \ N(\mathfrak{P}_i) = 2,$$

where $\mathfrak{P}_1$ corresponds to the divisor $2+\theta+\eta$, $\mathfrak{P}_2$ to $-1+\theta$, and $\mathfrak{P}_3$ to $1+\eta$.

The decomposition of other rational primes may be determined from the decomposition of $f(x)$ in prime functions (mod 2) as in the theorem of Dedekind.

The decomposition of 2 in Dedekind ideals obtained by Bachmann (loc. cit.) is the following:

$$[2] = \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3,\ \ N\mathfrak{p}_i = 2,$$

where

$$\mathfrak{p}_1 = [2,\theta,\eta],\quad \mathfrak{p}_2 = [2,\theta+1,\eta],\quad \mathfrak{p}_3 = [2,\theta,\eta+1].$$

It may be readily shown that the ideal $\mathfrak{p}_1$ consists of all integers of K divisible (mod 2) by $2+\theta+\eta$, $\mathfrak{p}_2$ of all integers divisible (mod 2) by $-1+\theta$, and $\mathfrak{p}_3$ of all integers divisible (mod 2) by $1+\eta$.

THE ORTHOPOLE LOCI OF SOME ONE-PARAMETER SYSTEMS OF LINES REFERRED TO A FIXED TRIANGLE

By O. J. RAMLER, The Catholic University of America

An orthopole is defined by J. Neuberg[1] as follows: Let there be a triangle ABC and a line L in the plane of the triangle. From A, B, and C draw perpendiculars to L cutting it in P, Q, and R, respectively. From P draw a perendicular to BC, from Q a perpendicular to AC, and from R a perpendicular to AB. These three perpendiculars meet in a point S which is called the orthopole of L with respect to the triangle ABC.

It is the purpose of this paper to determine and study the loci of orthopoles of some singly infinite systems of lines. A statement of some of the more important theorems to be found in the literature bearing on the subject is first given.

Theorem I[2]: *The orthocenter of a triangle is the orthopole of each side of the triangle referred to itself.*

Theorem II[2]: *The orthopole of a line perpendicular to a side of a triangle is the foot of that perpendicular.*

Theorems I and II follow immediately from the definition.

Theorem III: *The orthopoles of a system of parallel lines lie on a line perpendicular to that system.*

Theorem IV: (a) *The locus of orthopoles of a pencil of lines on a point P is a conic whose center is the midpoint of PH where H is the orthocenter of the reference triangle.*

(b) *If the point P is the circumcenter[3] of the reference triangle, the locus of orthopoles of the pencil of lines is the nine-point circle of the triangle.*

(c) *If P is on the circumcircle of ABC, and p is its pedal or Simson line referred to ABC, then the conic is the line p taken twice.*

Theorem V[4]: *The Simson lines of the extremities of any chord TT' of the circle ABC pass through the orthopole of TT'.*

Theorem VI[2]: *The locus of orthopoles of tangents to a curve of class m is a curve of order $2m$; the locus of lines whose orthopoles move on a curve of order n is a curve of class $2n$.*

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[1] J. Neuberg, *Sur une transformation des figures*, Nouvelle Correspondance Mathématique, vol. 4 (1878), p. 379.

[2] J. Neuberg, *Die Verwandtschaft zwischen einer Geraden und ihrem Lotpunkt in Bezug auf ein Dreieck*, Archiv der Mathematik und Physik, (3), vol. 3 (1902).

[3] M. Soons, Mathesis (1896), p. 57. K. Cwojdzinski, *Der Lotpunkt, ein neuer merkwürdiger Punkt des Dreiecks*, Archiv der Mathematik und Physik, (3), vol. 1 (1901), p. 175.

[4] Gallatly, *The Modern Geometry of the Triangle*, 2nd edition, Chapter VI, p. 49.

Neuberg has shown that the relation between a line m and its orthopole M is three to one; that is, a line m has a definite orthopole M, but a point M may be the orthopole of three distinct lines m_1, m_2, m_3. Three lines so related are said to be "associated." After showing that the vertices of the triangle $m_1m_2m_3$ (at least one of whose sides and the vertex opposite to it are real) lie on the circumcircle of the fundamental triangle ABC, Neuberg gives the following construction to find m_2 and m_3 when m_1 is known: Determine M as the orthopole of m_1 with respect to ABC; join the orthocenter H of ABC to M and produce it to H_m making $MH_m = HM$. Let m_1 meet the circumcircle of ABC in N_2 and N_3. Construct the triangle $N_1N_2N_3$ having H_m as orthocenter and ABC as circumcircle. This is done by drawing through N_2 a perpendicular to N_3H_m which meets the circle ABC again at N_1. Then N_1N_2 and N_1N_3 are the required lines m_3 and m_2, respectively. If m_1 does not meet the circle ABC, then the sides N_1N_2 and N_1N_3 are imaginary, meeting at the real vertex N_1, which is found by drawing N_1H_m perpendicular to m_1.

The theorems mentioned above and a number of additional theorems may be readily obtained by the use of "absolute" or "circular" coordinates.[5] Taking the vertices of our fundamental triangle ABC on a circle of unit radius and denoting the unit vectors from O, the circumcenter, to the vertices as α, β, γ, we call their symmetric functions $\sigma_1, \sigma_2, \sigma_3$. Let x be the vector to any moving point and y its conjugate. Then if the reflection of the origin O in any line m is u, we have as the conjugate equation of the line m

$$u^{-1}x + v^{-1}y = 1, \tag{a}$$

where v is the conjugate of u. The orthopole M of m with respect to ABC is then given by

$$x = \tfrac{1}{2}\sigma_1 + \tfrac{1}{2}u + \tfrac{1}{2}u^{-1}v\sigma_3. \tag{1}$$

To prove Theorem IV (b) we must put $u=v=0$. The product vu^{-1} is a unit vector, say t. The vector to the orthopole of a circumdiameter is therefore given by the equation

$$x = \tfrac{1}{2}(\sigma_1 + t),$$

which is the map equation of the nine-point circle.

Theorem I is proved by putting $u = \frac{1}{2}(\beta+\gamma)$, $v = (\beta+\gamma)/2\beta\gamma$ in (1), whence $x = \sigma_1$, the vector to the orthocenter of ABC. Theorem III is proved by putting $u^{-1}v = \text{constant} = k$. Equation (1) becomes $2x = k + u$, where u runs along the normal from O to the system of parallel lines. The equation $2x = k + u$ says that the locus of orthopoles is homothetic to this normal with respect to the fixed point given by k. Hence the locus is a straight line.

Considering the line m as the line through the points t_1 and t_2 on the circum-

[5] Winger, *Projective Geometry*, p. 324. F. Morley, Transactions of the American Mathematical Society, vol. 1 (1900), pp. 97–115; vol. 4 (1903), p. 1; vol. 8 (1907), pp. 14–24.

circle of ABC we obtain another and sometimes more useful expression for the vector to the orthopole, namely

$$x = \tfrac{1}{2}\sigma_1 + \tfrac{1}{2}(t_1 + t_2) + \tfrac{1}{2}\sigma_3 t_1^{-1} t_2^{-1}. \tag{2}$$

Now associate a third point t_3 on the unit circle with t_1 and t_2, and let s_1, s_2, and s_3 be their symmetric functions; equation (2) may then be written

$$x = \tfrac{1}{2}(\sigma_1 + s_1) - \tfrac{1}{2}t_3 + \tfrac{1}{2}\sigma_3 t_3 s_3^{-1}. \tag{3}$$

This gives the orthopole of t_1t_2. Now write the equation

$$x = \tfrac{1}{2}(\sigma_1 + s_1) - \tfrac{1}{2}(1 - \sigma_3 s_3^{-1})t, \tag{4}$$

where t is any point on the circumcircle of ABC. Giving t the values t_3, t_2, t_1, we get in turn the orthopoles of lines t_1t_2, t_3t_1, and t_2t_3, respectively. Equation (4) is the map equation of a circle, the circumcircle of the three orthopoles of the sides of the triangle $t_1t_2t_3$, referred to ABC. The center of this circle is at $\frac{1}{2}(\sigma_1+s_1)$, the midpoint of the line segment joining the orthocenter of ABC to that of $t_1t_2t_3$. The radius is one-half the absolute value of $1-\sigma_3 s_3^{-1}$. The symmetry of these results suggests

Theorem VII: *If the product of the vectors to the vertices of one triangle inscribed to the unit circle is equal to the product of the vectors to the vertices of a second inscribed triangle, the midpoint of the line joining their orthocenters is the orthopole of any side of either triangle referred to the other.*

The Deltoid

In 1857 Steiner[6] stated the theorem that the envelope of the Simson lines referred to a triangle is a curve of class three and order four. Later Cremona[7] showed that this curve is a hypocycloid of three cusps inscribed to the triangle. Morley[8] has named this curve the *deltoid*. We shall refer to it as the Steiner deltoid of the triangle in which it is inscribed.

Since the altitudes and sides of a triangle are Simson lines of the corresponding vertices and their reflections in the circumcenter, respectively, we have as a consequence of theorems I and II the following:

Theorem VIII: *The orthopole locus of the tangents to the Steiner deltoid of a triangle ABC is a quartic curve having a triple point at the orthocenter, and passing through the vertices of the pedal triangle. The three tangents at the triple point are parallel to the sides of the triangle.*

According to Theorem VI the locus should properly be a sextic, but since

[6] J. Steiner, *Über eine besondere Curve dritter Klasse (und vierten Grades)*, Crelle's Journal vol. 53 (1857), p. 231.

[7] M. Cremona, *Sur l'hypocycloide à trois rebroussements*, Crelle's Journal, vol. 64 (1865), p. 101.

[8] F. Morley, *Orthocentric properties of the plane N-line*, Transactions of the American Mathematical Society, vol. 4 (1903), p. 1.

the deltoid has the line at infinity as a double tangent, the locus degenerates into a quartic and the line at infinity taken twice.

The pedal or Simson line of a point $P(t)$ of the circumference ABC has an equation

$$2tx - 2\sigma_3 y + \sigma_3 t^{-1} + \sigma_2 - t^2 - \sigma_1 t = 0, \tag{5}$$

where t is the vector to the point P. Differentiating with respect to t we obtain the map equation of its envelope,

$$x = \tfrac{1}{2}\sigma_1 + t + \tfrac{1}{2}\sigma_3 t^{-2}. \tag{6}$$

Equation (6) is then the map equation of the Steiner deltoid of ABC. Now write the equation,

$$x = \tfrac{1}{2}\sigma_1 + \tfrac{1}{2}(t_1 + t) + \tfrac{1}{2}\sigma_3 t^{-1}t_1^{-1}.$$

It is the equation of a two cusped hypocycloid, i.e., a line segment of length equal to that of the circumdiameter of ABC. Moreover this line segment meets the deltoid (6) where $t=t_1$ and has at that point the same direction as the deltoid. The segment is therefore tangent to the deltoid. If we put $t=t_2$ in (7) we again obtain equation (2) and x is then the vector to the point of intersection M of the tangents to the deltoid given by $t=t_1$ and $t=t_2$. Since these tangents are the Simson lines of ABC for the points t_1 and t_2 we have a proof of Theorem V. The deltoid being of class three, three tangents may be drawn to it from M. Let t_3 be the parameter giving the third tangent from M. Then triangle $t_1t_2t_3$ is such that its sides are the three lines each of which has M as its orthopole with respect to ABC. Whence we have:

Theorem IX: *The three lines that have a given point M as their orthopole with respect to a triangle ABC are real when M lies inside the Steiner deltoid of ABC; two of the three lines become coincident when M lies on the deltoid and but one line is real when M lies outside the deltoid. The coincidence of two of the three lines also requires the coincidence of two of the three parameters $t_1t_2t_3$.* Hence we have the:

Corollary: *The locus of the orthopoles of the tangents to the circumcircle of ABC is the Steiner deltoid of ABC.*

This result could easily have been obtained from equation (1) by putting $u=2t$, $v=2/t$. We merely remark here that the triangle $t_1t_2t_3$ also has a Steiner deltoid whose size and orientation are the same as those of the Steiner deltoid of ABC, and which is obtained by merely translating the latter curve by an amount equal to and in the direction of the vector from H to M, where H is the orthocenter of ABC.

We turn our attention next to Theorem IV (a) which was first stated by Neuberg.[9] Cwojdzinski,[10] without proof, states the following:

[9] Archiv der Mathematik und Physik, (3), vol. 3 (1902), p. 89.

[10] Archiv der Mathematik und Physik, (3), vol. 1 (1901), p. 178.

Theorem X: *If the lengths of the perpendiculars dropped from any point P upon the sides of ABC are laid off from the orthocenter on the upper segments of the altitudes, then the three points thus obtained together with the feet of the perpendiculars from P will lie on an ellipse whose center bisects the line segment joining the orthocenter of ABC to P.*

We have no way of knowing from Cwojdzinski's article that he was aware of the facts as stated in Theorem IV (a), particularly since Neuberg did not publish his theorem until the following year. Neuberg merely states that the locus of orthopoles of lines on a point P is a conic, but he does not give the type of conic. We now prove:

Theorem XI: *The locus of orthopoles of the lines on a fixed point with respect to a fixed triangle is an ellipse.* This includes Cwojdzinski's theorem.

Letting the vector to the fixed point P be z, and z' its conjugate, the equation of any line through P will be

$$x - z = t(y - z') ,$$

whence

$$u = z - tz', \quad v = (tz' - z)/t, \quad v/u = -1/t.$$

Hence the orthopole of any line of the pencil given by parameter t is

$$x = \tfrac{1}{2}(\sigma_1 + z) - \tfrac{1}{2}(tz' + \sigma_3 t^{-1}) \tag{8}$$

which is the map equation of an ellipse, because the elimination of t from (8) and its conjugate yields an equation of the second degree in x and y, and for no value of t as it moves on the unit circle can x become infinite. The major and minor axes of the ellipse are $1+|z'|$ and $|(1-|z'|)|$, respectively; that is if r is the circumradius of ABC, the major and minor axes are $r \pm OP$. The center is at $\frac{1}{2}(\sigma_1+z)$, that is, at the midpoint of HP, where H is the orthocenter of ABC. Putting $t=\beta\gamma$ in (8), we obtain the foot of the perpendicular from P upon BC. This point is therefore on the ellipse. Putting $t=-\beta\gamma$ we readily verify the second part of Theorem X. Again putting $z=t_1$ we obtain the locus of orthopoles of lines on a point of the circumcircle of ABC, namely,

$$x = \tfrac{1}{2}(\sigma_1 + t_1) - \tfrac{1}{2}(t_1^{-1} + \sigma_3 t^{-1}),$$

which is a two-cusped hypocycloid, or since $1-|t_1|=0$, an ellipse with zero minor axis; we have thus proved Theorem IV (c). The foci of the ellipse (8) are given by $dx/dt=0$, namely by $t=\pm(\sigma_3/z')^{1/2}$. The vectors to the foci are then

$$\tfrac{1}{2}(\sigma_1 + z) \pm (\sigma_3 z')^{1/2}.$$

Interpreting the above analysis geometrically we have the following scheme for the *determination of the directions of the axes of the ellipse locus*:

As axis of reals choose some line as OA. Then $\sigma_3^{1/2}$ is the midpoint of arc BC; call it X. Bisect angle AOP, letting the bisector cut the circumcircle at Q. Locate Y on the circumcircle so that angle $AOY = AOX - AOQ$. Then OY is parallel to the major axis.

The Inscribed Ellipse

The ellipse (8) intersects the side BC in points whose parameters are $t=\beta\gamma$ and $t=\alpha(z+\alpha)/(1+\alpha z')$, the former giving the foot of the perpendicular from P upon BC, the latter giving the second intersection which may be determined as the orthopole of a line through P parallel to OM, where M is the midpoint of AP. Equating the expressions for the two intersections we shall have a locus for P which will give ellipses tangent to BC. The locus is found to be the straight line,

$$z - \beta\gamma z' = \beta\gamma\alpha^{-1} - \alpha, \tag{9}$$

whose equation is satisfied by $z=-\sigma_1$ and $z=-\alpha$. Moreover the line (9) is perpendicular to BC; in fact it is the homothetic with respect to the circumcenter of the altitude upon BC. From the symmetry of the result, $z=-\sigma_1$, we have

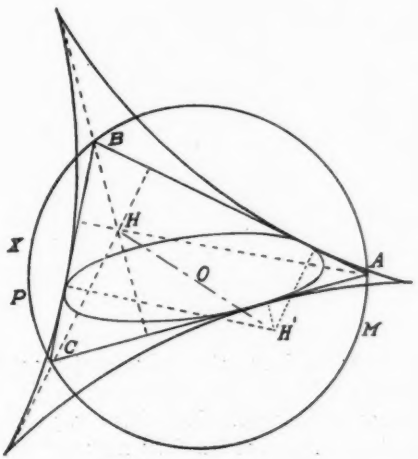

Fig. 1

Theorem XII: *The locus of orthopoles, with respect to a triangle, of lines on the point symmetric to the orthocenter with respect to the circumcenter is an ellipse tangent to the three sides of the triangle and concentric with the circumcircle.*

If the triangle is acute angled the ellipse will be inscribed; if the triangle is a right triangle, the ellipse is merely the hypotenuse taken twice; if the triangle is

obtuse, the ellipse is "escribed," lying within the obtuse angle. The points of tangency are the same as the points of tangency of the Steiner deltoid with the triangle, and are in fact the feet of the perpendiculars dropped from H' upon the sides, where H' is the symmetric of the orthocenter with respect to the circumcenter (Fig. 1).

For the sake of brevity some results are now given that can easily be determined by further investigation using methods similar to those used above.

The loci of orthopoles, referred to ABC, of pencils of lines on the four equicenters $I_i (i=1, 2, 3, 4)$ of ABC are four mutually tangent ellipses, whose centers E_i are members of an orthocentric set of four points, and whose com-

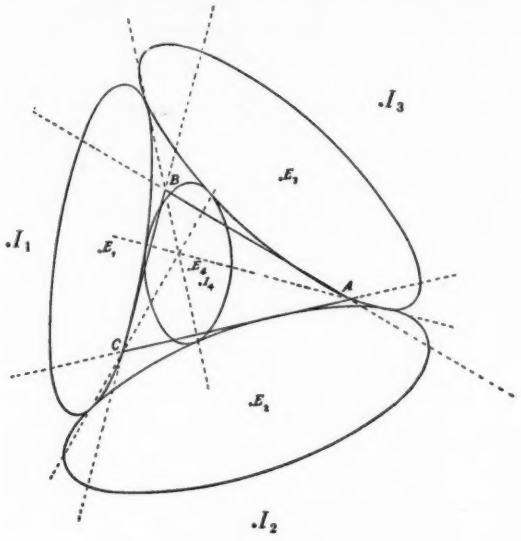

FIG. 2

mon tangents meet by threes in four points of an orthocentric set T_i. The four triangles contained in the set T_i and the reference triangle ABC have the same nine-point circle. The medial triangle of ABC is the pedal triangle to each of the four triangles contained in the set T_i. The six points of common tangency of the four ellipses lie in pairs on the altitudes of ABC. The six common tangent lines of the four ellipses are likewise tangent lines to the Steiner deltoid of ABC, the points of tangency being identical with the points of tangency of the six lines with the four ellipses.

MAXIMUM NUMBERS ASSOCIATED WITH THE DIOPHANTINE EQUATION $\Sigma(1/x_1x_2\cdots x_{n-1})=b/[(m+1)b-1]$

By H. A. SIMMONS, Northwestern University

1. *Introduction.* The case $r=n-1$ of the cyclo-symmetric equation which the author has recently studied[1] may be written in the form

$$\Sigma\frac{1}{x_1x_2\cdots x_{n-1}}=\frac{1}{(m+1)b-1}, \tag{1}$$

where Σ is the elementary symmetric function of the n variables $x_1, x_2, \cdots, x_n$, taken $(n-1)$ at a time, and where b, m are positive integers. The principal results obtained in the present paper are that the number v_n in the solution $(x_1, x_2, \cdots, x_n)\equiv(v_1, v_2, \cdots, v_n)$ of equation (1), where

$$v_i=1(i=1,2,\cdots,n-2),\quad v_{n-1}=m+1,\quad v_n=[(m+1)b-1](m+n-1) \tag{2}$$

is the largest number that exists in any solution in positive integers of this equation and that $v_1+v_2+\cdots+v_n$, $v_1v_2\cdots v_n$ are the largest sum and product, respectively, of the numbers in any solution of (1). These results constitute a partial solution of problems 1, 2, 3 which were stated at the end of the paper referred to above. Unfortunately the method which we use to identify v_n as the maximum number does not seem to apply except when $r=n-1$.

In §5 we exhibit for the equation $\Sigma(1/x_1x_2\cdots x_r)=b/[(m+1)b-1]$, in which r is a positive integer $<n$, a solution in positive integers which includes solution (2) of equation (1).

Throughout this paper $x_1, x_2, \cdots, x_{n-1}$ stand for positive integers, conveniently taken in the order $x_1\leqq x_2\leqq\cdots\leqq x_{n-1}$, while x_n, $(\geqq x_{n-1})$, in §§2, 3, 4, is not restricted to be an integer; in these sections the term *solution* means a set $(x_1, x_2, \cdots, x_n)$, occasionally referred to as *set* x, which satisfies equation (1). In each of the three problems which we solve, x_n assumes an integral value.

2. *An auxiliary theorem.* In making the proof proposed above, we shall need Theorem 1 below, in which the variables $u_1, u_2, \cdots, u_n$ are positive integers, taken in the order $u_1\leqq u_2\leqq\cdots\leqq u_n$.

Theorem 1a: $u_1+u_2+\cdots+u_n\leqq u_1u_2\cdots u_n+n-1$.

Theorem 1b: *The equality sign in* 1a *holds if, and only if,* $u_i=1(i=1, 2, \cdots, n-1)$.

Proof of 1a: If $n=1$, Theorem 1a is obviously true. If $n=2$, the inequality also holds since $u_1+u_2\leqq u_1u_2+1$ is equivalent to $(u_1-1)(u_2-1)\geqq 0$. Now if

$$u_1+u_2+\cdots+u_k\leqq u_1u_2\cdots u_k+k-1,$$

then

[1] This Monthly, vol. 36 (1929), pp. 148–155.

$$u_1 + u_2 + \cdots + u_k + u_{k+1} \leqq u_1u_2 \cdots u_k + u_{k+1} + k - 1 \leqq u_1u_2 \cdots u_{k+1} + k,$$

or

$$u_1 + u_2 + \cdots + u_{k+1} \leqq u_1u_2 \cdots u_{k+1} + k\ ;$$

hence Theorem 1a is true for all positive integral values of n.

Proof of 1b: Suppose $u_i = 1 (i = 1, 2, \cdots, n-1)$. Then the equality sign in Theorem 1a holds whatever be the value of u_n. We wish to prove that the inequality sign in Theorem 1a holds if two or more of the u's exceed 1, so that certainly $u_{n-1} > 1$, $u_n > 1$. With this hypothesis, $(u_n - 1)(u_{n-1} - 1) > 0$, so that $u_n + u_{n-1} < u_nu_{n-1} + 1$. Then by induction of the type used above, we find that $u_1 + u_2 + \cdots + u_n < u_1u_2 \cdots u_n + n - 1$; hence Theorem 1b is true.

Incidentally we state for Theorem 1 a generalization which can be obtained by induction.

Theorem 2a: $\Sigma u_1u_2 \cdots u_r \leqq {}_{n-1}C_{r-1} \cdot u_1u_2 \cdots u_n + {}_{n-1}C_r$.

Theorem 2b: *The equality sign in Theorem* 2a *holds if, and only if,* $u_i = 1 (i = 1, 2, \cdots, n-1)$.

Putting $u_i = 1 (i = 1, 2, \cdots, n)$ in Theorem 2a, we obtain as a corollary the well known relation:

$$ {}_nC_r = {}_{n-1}C_{r-1} + {}_{n-1}C_r.$$

3. *Proof that the* v_n *of* (2) *is the maximum number in a solution of* (1). It is convenient to consider (1) in the form

$$\frac{1}{x_1x_2 \cdots x_{n-1}} + \frac{1}{x_n} \cdot \frac{x_1 + x_2 + \cdots + x_{n-1}}{x_1x_2 \cdots x_{n-1}} = \frac{b}{(m+1)b - 1}. \tag{3}$$

Let $X_1, X_2, \cdots, X_n$ denote a solution in which X_n is the sought maximum number. On solving (3) for x_n and introducing X in place of x, we obtain

$$X_n = \frac{[(m+1)b - 1](X_1 + X_2 + \cdots + X_{n-1})}{bX_1X_2 \cdots X_{n-1} - (m+1)b + 1},$$

in which the denominator is a positive integer. Consequently,

$$X_n \leqq [(m+1)b - 1](X_1 + X_2 \cdots + X_{n-1}).$$

Application of Theorem 1a gives

$$X_n \leqq [(m+1)b - 1](X_1X_2 \cdots X_{n-1} + n - 2). \tag{4}$$

From (2) we also have

$$X_n \geqq v_n. \tag{5}$$

From (3), (5), and Theorem 1a, we now observe that

$$\frac{1}{X_1X_2 \cdots X_{n-1}} + \frac{X_1X_2 \cdots X_{n-1} + n - 2}{v_n \cdot X_1X_2 \cdots X_{n-1}} \geqq \frac{b}{(m+1)b - 1},$$

from which

$$(6)\quad X_1X_2\cdots X_{n-1} \leqq \frac{v_n+n-2}{b(m+n-1)-1} \equiv \frac{[(m+1)b-1](m+n-1)+n-2}{b(m+n-1)-1} = m+1.$$

From (3) it is obvious that $X_1X_2\cdots X_{n-1}\geqq m+1$. Then, on account of (6),

$$(7)\qquad X_1X_2\cdots X_{n-1} = m+1.$$

From (4) and (7), therefore, $x_n\leqq v_n$, which, in connection with (5), proves that $x_n=v_n$.

From Theorem 1b and the fact that $X_1+X_2+\cdots+X_{n-1}=X_1X_2\cdots X_{n-1}+n-2=m+n-1$, it is evident that the maximum number v_n appears in no solution of (1) except (2).

Remark: Solution (2) is not necessarily the only solution of (1). For example, if $n=4$, $b=1$, $m=10$, equation (3) becomes

$$1/x_1x_2x_3 + (x_1+x_2+x_3)/x_1x_2x_3x_4 = 1/10,$$

which has (perhaps among others) the solutions (1, 1, 12, 70), (1, 1, 15, 34), (1, 1, 20, 22) in addition to solution (2), which is (1, 1, 11, 130). There may also be more than one solution when $b>1$. For example, if $n=5$, $b=2$, $m=2$, equation (3) becomes

$$1/x_1x_2x_3x_4 + (x_1+x_2+x_3+x_4)/x_1x_2x_3x_4x_5 = 2/5,$$

which has (perhaps among others) the solutions (1, 1, 2, 3, 5), (1, 1, 1, 5, 8), (1, 1, 2, 2, 10) in addition to solution (2), which is (1, 1, 1, 3, 30).

We summarize our result in

Theorem 3: *The maximum number that exists in a solution of the equation* $\Sigma(1/x_1x_2\cdots x_{n-1})=b/[(m+1)b-1]$, in which b, m are positive integers is $[(m+1)b-1](m+n-1)$. This number appears in but one solution, namely solution (2): $v_i=1 (i=1, 2, \cdots, n-2), v_{n-1}=m+1, v_n=[(m+1)b-1](m+n-1)$.

4. *The maximum sum and product of the numbers in a solution of* (1). From (3) it is obvious that the maximum sum and the maximum product are simultaneously attained. We shall solve both problems by finding the solution $(x_1, x_2, \cdots, x_n)\equiv(Y_1, Y_2, \cdots, Y_n)$ of maximum sum. Our proof follows from the set of lemmas below, in which S stands for the maximum sum, $Y_1+Y_2+\cdots+Y_n$.

Lemma 1: $S\geqq b(m+1)(m+n-1)$.

This follows from the definition of S and the fact that the sum of the numbers in solution (2) equals $b(m+1)(m+n-1)$.

Lemma 2: $Y_{n-1}\leqq m+1$.

Proof: Suppose $Y_{n-1}>m+1$, say $Y_{n-1}=m+1+\delta$, where δ is a positive in-

teger, and $Y_{n-1} \leqq Y_n$ since the set Y is a set x. We shall reach a contradiction by showing that if α is a positive integer $\leqq \delta$, so that

$$b/[(m+1)b-1] > 1/Y_1Y_2 \cdots Y_{n-2}(Y_{n-1}-\alpha)$$

whatever be the (positive integral) value of $Y_1Y_2 \cdots Y_{n-2}$, and if α, Y_n' are such that the set $Y_1, Y_2, \cdots, Y_{n-2}, Y_n'-1 \equiv Y_{n-1}-\alpha, Y_n'$ satisfies (1), then $Y_{n-1}'+Y_n' > Y_{n-1}+Y_n$, and the set Y does not have the largest sum possessed by any solution of the type that we consider.

Letting $\pi = Y_1Y_2 \cdots Y_{n-2}$ and $\sigma = Y_1+Y_2+\cdots+Y_{n-2}$, we have, by hypothesis [see (3)],

$$(8_1) \qquad \frac{1}{Y_{n-1}}+\frac{1}{Y_n}+\frac{\sigma}{Y_nY_{n-1}} = \frac{b\pi}{a},$$

$$(8_2) \qquad \frac{1}{Y_{n-1}'}+\frac{1}{Y_n'}+\frac{\sigma}{Y_n'Y_{n-1}'} = \frac{b\pi}{a}.$$

The desired result will now follow from (8_2) and the definitions of Y_n', $Y_n'-1$ if we can show that

$$(9) \qquad \frac{1}{Y_{n-1}-\alpha}+\frac{1}{Y_n+\alpha}+\frac{\sigma}{(Y_{n-1}-\alpha)(Y_n+\alpha)} > \frac{b\pi}{a}.$$

That (9) follows from (8_1) is a consequence of the two equivalent statements,

$$\frac{1}{Y_{n-1}-\alpha}+\frac{1}{Y_n+\alpha} > \frac{1}{Y_{n-1}}+\frac{1}{Y_n} \quad \text{and} \quad \frac{\sigma}{(Y_{n-1}-\alpha)(Y_n+\alpha)} > \frac{\sigma}{Y_nY_{n-1}}.$$

Hence Lemma 2 is true.

Lemma 3: $Y_{n-1} \geqq m+1$.

Suppose $Y_{n-1} < m+1$, say $Y_{n-1} = m+1-\delta$, where δ is a positive integer, $\leqq m-1$ since $Y_{n-1} \geqq 2$. Then $Y_n < a(m+n-1)$, [see (2) and Theorem 3]. Therefore $Y_1+Y_2+\cdots+Y_{n-1} \geqq m+n$ by Lemma 1. To make the desired proof now, we only need to establish two facts: (p) if $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ are positive integers such that $\alpha_1+\alpha_2+\cdots+\alpha_{n-1} \geqq m+n-1$, then $\alpha_1\alpha_2 \cdots \alpha_{n-1} \geqq m+1$; ($q$) if a set $\beta_1, \beta_2, \cdots, \beta_n$ is a set x for which $\beta_1+\beta_2+\cdots+\beta_{n-1} \geqq m+n$, then the set β is not the set Y.

Proof of (p): This is an immediate consequence of Theorem 1a.

Proof of (q): Consider the set $\beta_1, \beta_2, \cdots, \beta_{n-2}, \beta_n'-1 \equiv \beta_{n-1}-1, \beta_n'$, in which β_n' is so selected that these n numbers form a solution of (1). When the first $n-1$ elements of this set are suitably arranged, a set x results for which $\beta_1\beta_2 \cdots \beta_{n-2}\beta_n'-1 \geqq m+1$, by ($p$); and such that $\beta_n'-1+\beta_n' > \beta_{n-1}+\beta_n$, as can be shown by the type of argument that was used in proving Lemma 2. Hence (q), and consequently Lemma 3, is true.

From the last two lemmas, $Y_{n-1} = m+1$. By the method used in proving

Lemma 2, one can now readily show that $Y_i=1 (i=1, 2, \cdots, n-2)$. Hence the set Y is identical with solution (2).

We state the results of this section in:

Theorem 4. *The sum and the product of the numbers in solution* (2) *are greater than the sum and product, respectively, of the numbers in any other solution of equation* (1).

5. *A solution of* $\Sigma(1/x_1x_2\cdots x_r)=b/[(m+1)b-1]$. By such an inductive attack as was employed to obtain a solution of the cyclo-symmetric equation mentioned above, we have found for the equation

$$\sum \frac{1}{x_1x_2\cdots x_r} = \frac{b}{(m+1)b-1}, \tag{10}$$

a solution which includes solution (2) of equation (1). In order to exhibit this solution, we define the symbol $\Sigma_{i,j}$ as the elementary symmetric function of $x_1, x_2, \cdots, x_i$, $(i<n)$, taken $j \leqq i$ at a time. In terms of this symbol it is not difficult to see that (10) can be written in the form

$$\frac{1}{x_1x_2\cdots x_r} + \frac{1}{x_{r+1}}\cdot\frac{\Sigma_{r,1}}{x_1x_2\cdots x_r} + \frac{1}{x_{r+2}}\cdot\frac{\Sigma_{r+1,2}}{x_1x_2\cdots x_{r+1}} + \cdots + \frac{1}{x_n}\cdot\frac{\Sigma_{n-1\ n-r}}{x_1x_2\cdots x_{n-1}} = \frac{b}{a}, \quad a \equiv (m+1)b-1. \tag{11}$$

When $n=r$, (11) does not always have a solution in positive integers. For example, if $n=r=3$, $b=2$, $m=2$, equation (11) becomes $1/x_1x_2x_3=2/5$, which has no such solution. When $n>r$, a solution of (11) is given by the numbers $(x_1, x_2, \cdots, x_n)=(w_1, w_2, \cdots, w_n)$, where

$$\begin{aligned}
&(12a) \quad w_k = 1 \ (k = 1,2, \cdots, r-1) ; \\
&(12b) \quad w_r = m+1 ; \\
&(12c) \quad w_{k+1} = a\textstyle\sum_{k,k-r+1} + 1 \ (k = r, r+1, \cdots, n-2) ; \\
&(12d) \quad w_n = a\textstyle\sum_{n-1,n-r} .
\end{aligned} \tag{12}$$

To see that (12) is a solution of (11), substitute into (11) for the x's their values from (12); multiply the resulting equation through by a; in the result add 1 to, and subtract 1 from each numerator after the first; then collect the first two terms, using $(m+1)b-1$ in place of a, and get $b-1/(m+1)(a\Sigma_{r,1}+1)$; next, to the result just obtained, add the third term and get

$$b-1/(m+1)(a\Sigma_{r,1}+1)(a\Sigma_{r+1,2}+1);$$

to this result add the fourth term and get

$$b-1/(m+1)(a\Sigma_{r,1}+1)(a\Sigma_{r+1,2}+1)(a\Sigma_{r+2,3}+1); \text{ etc.}$$

By the methods used in connection with the cyclo-symmetric equation, we have proved for solution (12) properties that are analogous to those which were found for our solution of the cyclo-symmetric equation.

We therefore wish to propose to the reader problems which are analogous to those that were proposed in connection with the cyclo-symmetric equation.

Problem 1: If $n>r$, is the w_n of (12) the largest number that exists in a solution in positive integers of equation (10)?

Problem 2: If $n>r$, are $w_1+w_2+\cdots+w_n$ and $w_1w_2\cdots w_n$ the maximum sum and product, respectively, of any set of n positive integers which constitute a solution of equation (10)?

ON A POINT-TO-PLANE TRANSFORMATION IN FOUR-SPACE AND AN ∞^4-SYSTEM OF TETRAHEDRAL COMPLEXES IN ANY THREE-SPACE CONTAINED IN THE FOUR-SPACE

By B. C. WONG, University of California at Berkeley

Consider two hyperquadric surfaces Q and Q' in 4-space whose equations referred to their common self-polar simplex are, respectively,

$$\sum_{i=0}^{4} a_i x_i = 0, \qquad \sum_{i=0}^{4} a_i' x_i = 0.$$

The polar hyperplanes of a given point $P(y)$ with respect to Q and Q' intersect in a plane π given by the equations,

$$\sum a_i y_i x_i = 0, \qquad \sum a_i' y_i x_i = 0.$$

The plane π is said to correspond to the point P. This paper proposes to describe some of the features of this point-to-plane transformation and also to obtain a system of ∞^4 tetrahedral complexes contained in any 3-space of S_4 in connection with the transformation. It will be found that a general line determines a quadric hypercone of planes, a general plane a cubic surface which is peculiarly related to it, and a general hyperplane a normal quartic curve which in turn determines a unique twisted cubic in the hyperplane. We shall also find that a general hyperplane section of all the planes obtained in this transformation consists of all the ∞^4 lines of the hyperplane forming an ∞^4-system of tetrahedral complexes.

From the equations of transformation above or otherwise we see that the vertices of the self-polar simplex go into the opposite hyperplanes or faces, any point on an edge into the opposite plane, any point in a plane into a plane through the opposite edge, and any point in a face into a plane through the opposite vertex.

Let a point $P(y)$ describe a line l given by the intersection of three hyperplanes $S_3(u)$, $S_3'(u')$, $S_3''(u'')$. The corresponding plane π describes a quadric hypercone V_3^2 whose equation is

$$\Delta \equiv \begin{vmatrix} a_0x_0 & a_1x_1 & a_2x_2 & a_3x_3 & a_4x_4 \\ a_0'x_0 & a_1'x_2 & a_2'x_2 & a_3'x_3 & a_4'x_4 \\ u_0 & u_1 & u_2 & u_3 & u_4 \\ u_0' & u_1' & u_2' & u_3' & u_4' \\ u_0'' & u_1'' & u_2'' & u_3'' & u_4'' \end{vmatrix} = 0.$$

Consider for a moment the hyperplane $S_3(u)$. Its points transform into the ∞^3 trisecant planes of a definite 4-space quartic curve C^4. The equations of C^4 can be obtained by equating to zero the matrix of the first three rows of Δ or can be written parametrically

$$\rho x_i = u_i/(a_i - \lambda a_i') \qquad [i = 0,1, \cdots, 4].$$

To a given S_3 there is a definite C^4 corresponding and since there are ∞^4 hyperplanes in S_4, there are ∞^4 such quartic curves. They all have in common the five vertices of the self-polar simplex. The quadric hypercone V_3^2, and there are ∞^6 such corresponding to the ∞^6 lines in S_4, has on it ∞^2 quartics of the type C^4 and through each C^4 pass ∞^4 hypercones of the type V_3^2.

Each C^4 is also the locus of points whose corresponding planes all lie in the S_3 determining C^4. The planes are such that through a general point P of S_3 three of them pass. These three planes come from the three points on C^4 determining the plane π corresponding to P. Hence the planes in question envelop a twisted cubic curve γ^3 whose points go into the osculating planes of C^4. S_3 meets C^4 in four points forming a tetrahedron. The cubic γ^3 osculates the four faces of this tetrahedron and also the five planes in which S_3 intersects the five hyperplanes of the self-polar simplex.

The curve C^4 is also the locus of the poles of S_3 with respect to all the hyperquadric surfaces of the pencil determined by Q and Q'.

Now let P describe a plane ϕ common to $S_3(u)$ and $S_3'(u')$. The locus of points whose corresponding planes meet ϕ in lines is a ruled cubic surface in S_4 and has for equations the matrix of the first four rows of Δ equated to zero. This cubic surface, F^3, is also the locus of the vertices of the ∞^2 hypercones corresponding to the ∞^2 lines in ϕ. If ϕ is a plane π' corresponding to a point P', that is, an osculating plane of γ^3 in S_3, F^3 becomes a point coincident with P' which is on C^4. It is not difficult to see that a point P in ϕ goes into a plane π meeting F^3 in a conic, for a 3-space through P meets F^3 in a twisted cubic curve and π in a line bisecant to the curve.[1] There are ∞^2 conics on F^3 corresponding

[1] The line is said to correspond to the point P in a point-to-line transformation by means of two quadric surfaces in the 3-space. See Bing Chin Wong, *A study and classification of ruled quartic surfaces by means of a point-to-line transformation*, University of California Publications in Mathematics, vol. 1, No. 17, pp. 371–387.

to the ∞^2 points in ϕ. If P describes a line l in ϕ, there is an ∞^1-system of conics on F^3 all having a point L in common and their planes are the planes of the V_3^2 corresponding to l and having L for vertex.

While through a given C^4 pass ∞^3 F^3 corresponding to the ∞^3 planes in S_3, on each F^3 lie ∞^1 C^4 corresponding to the ∞^1 hyperplanes through ϕ. F^3 may be said to be generated by this ∞^1-system of C^4. If we consider the points of F^3 and the lines of ϕ as corresponding elements, we have in ϕ an ∞^1-family of curves of class 3 and order 4, each curve being the intersection of ϕ with the developable to the γ^3 contained in an S_3 through ϕ. The tangents to such a curve correspond to the points of the C^4 on F^3 determined by S_3. All the curves of the family have five common tangents which are the lines of intersection of ϕ with the faces of the self-polar simplex.

Consider ϕ as a plane in a fixed S_3. Any hyperplane S_3', which may be S_3 itself, meets the F^3 corresponding to ϕ in a cubic curve C^3 and determines a quartic curve C'^4. Let S_3 meet C'^4 in the four points A, B, C, D. The points of C^3 go into lines enveloping a conic κ^2 in ϕ. If C^3 is made up of a line g and a conic C^2, κ^2 degenerates into two pencils of lines. In this case, one of the four points common to S_3 and C'^4, say A, is in ϕ and the vertex of one of the pencils of lines in ϕ is at A, the vertex of the other is on the line in which ϕ meets the plane BCD. Now if C^3 is made up of three lines of which one, g, meets the other two, g', g'', in distinct points, two of the four points common to S_3 and C'^4, say A and B, are in ϕ and they are the vertices of the two pencils making up the degenerate κ^2. Corresponding to the points on g which must be the line CD is the same line AB. But if the components g, g', g'' of C^3 are concurrent at a point P' of F^3, three of the four points A, B, C, D, say A, B, C, are in ϕ which is now coincident with an osculating plane π' of the γ^3 in S_3 and the point P' is no other than the point D itself.

Note that there is another correspondence between the points of a plane θ, which may be coincident with ϕ, and the planes of the conics of F^3. This correspondence arises from associating with a given point P' of θ that plane π which passes through P' and meets F^3 in a conic. As π is the transform of a point P in ϕ, we have a quadratic birational transformation between the points of ϕ and those of θ. If P describes a line l in ϕ, P' describes a conic which is the intersection of θ with with the V_3^2 corresponding to l; and if P describes a conic in ϕ, the corresponding plane π describes a V_3^4 containing F^3 as double surface, thus giving rise to a trinodal quartic curve in θ as the locus of P'. But if P' describes a line in θ, π describes a V_3^4 containing F^3 doubly and hence P in ϕ describes a conic. This correspondence between the points of θ and ϕ is, therefore, birational and quadratic.

Now we come to the consideration of the tetrahedral complexes. It is known that the ∞^3 trisecant planes of a normal quartic curve in S_4 meet any S_3 in the lines of a tetrahedral complex, the fundamental tetrahedron being formed by

the points in which S_3 meets the curve.[2] Since the points of every hyperplane transform into the trisecant planes of a quartic curve which that hyperplane determines, there are ∞^4 tetrahedral complexes in a given S_3 obtained in this way. Each of the ∞^4 tetrahedra has for faces four of the osculating planes of the unique cubic γ^3 contained in S_3. Any four osculating planes of γ^3 form such a tetrahedron. Among the complexes there are many degenerate ones. The nature of the degeneracy of such a complex depends upon the relation which the corresponding hyperplane bears to the simplex self-polar to Q and Q', such as passing through one, two, three, or even four of its vertices. We shall not go into this.

Any two complexes Γ', Γ'' whose lines lie in the planes coming from the points of two hyperplanes S_3', S_3'' respectively have in common a congruence of lines bisecant to a cubic curve C^3. The lines of this congruence lie in the planes which are trisecant to the two quartic curves C'^4, C''^4 determined by S_3', S_3'' respectively. C^3 is the intersection of S_3 with the F^3 determined by the plane ϕ common to S_3' and S_3'', and it passes through the vertices of the two tetrahedra of Γ' and Γ'' and indeed through all the vertices of the ∞^1 tetrahedra whose complexes correspond to the ∞^1 hyperplanes through ϕ. There are ∞^6 such cubics giving rise to as many congruences corresponding to the ∞^6 planes in S_4. Any two congruences if they belong to the same complex have a quadratic regulus in common and if they belong to different complexes have just a line in common. Any three complexes always have in common a regulus which may be a proper quadric surface, or a quadric cone, or a pair of pencils of planes.

RECENT PUBLICATIONS

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All books for review should be sent directly to the editor of this department and not to any of the other editors or officers of the Association.

REVIEWS

Einführung in die mathematische Behandlung naturwissenschaftlicher Fragen. Erster Teil: Funktion und Graphische Darstellung. Differential- und Integral Rechnung. By Alwin Walther. Julius Springer, Berlin, 1928. viii+220 pages.

The author informs us in his preface that this volume, entitled "Introduction to the mathematical treatment of questions in natural science," is a result of a course which he gave in the years 1924–1926 at Göttingen, primarily for physicists, chemists and biologists. The author's objective, he informs us, was not to load the reader with formulae nor to give him worthless skill in mere

[2] Carrone, *Sopra un nuovo modo di generazione del complesso tetraedrale.* Napoli Rendiconti (3), No. 7, pp. 57–66.

manipulation, but rather to give a clear understanding of mathematical methods in their application to questions of natural science.

The volume is divided into two sections. Section I, pages 1–67, is entitled "Function and Graphical Representation." Section II, pages 67–202, is entitled "Differential and Integral Calculus."

The first 67 pages are devoted to a treatment of functions and graphs. While the principal consideration is of continuous, one valued functions and of functions whose corresponding graphs have only a finite number of oscillations within a finite interval, attention is directed to the "natural" existence of discontinuous functions of various kinds as well as to functions whose corresponding graphs have an infinite number of oscillations within a finite interval. The treatment of functions and graphs, though very brief, introduces rectangular coordinates, polar coordinates, logarithmic, semi-logarithmic and triangulation graphing and their uses as a means of adequate mathematical formulation from empirical data, in the various fields of natural science. Even the uses of nomographic scales for representation of various functions of several variables is given a brief treatment. If we note that the author includes a treatment of interpolation, deriving Newton's as well as Stirling's formulae, it is evident that these 67 pages are concentrated.

The next 135 pages are devoted to what constitutes the basis of the usual first course in differential and integral calculus in most of our colleges. The following topical heads will indicate, to some extent, the author's treatment of the subject:

Pages 67–79: *Derivative and indefinite integral.* Integration treated as an inverse operation, differentiation of functions of several variables, partial differentiation; applications of integration.

Pages 79–91: *Applications of the first and higher derivatives.*

Pages 91–117: *The differential and increment of a function.* Infinitesimals with applications; theorem of the mean; Taylor's series with remainder; Taylor's infinite series; brief mention of Fourier series and their uses.

Pages 117–138: *The definite integral.* Integration as a limit of a sum of infinitesimals; mechanical integration (planimeter); graphical integration.

Pages 138–165: *Derivation of formulae for differentiation and integration.* Additional applications.

Pages 165–202: *The natural logarithm and exponential function.* The differential and integral calculus of such functions in problems that arise in natural science.

The volume has 174 figures, which indicates the author's emphasis on the need and value of visualizing the significance of the corresponding analysis (granting that often in special considerations the analysis transcends such possibility—as in continuous functions without derivatives at any point).

This volume is not a treatise nor is it a text book. It does not contain drill exercises. The greatest value of this volume is not in the advancement of the content of mathematics but in advancing the appreciation of mathematics as

a marvelous tool in the heads and hands of the natural scientists. The book should prove a stimulus to the scientist whose work has emerged from the empirical to the analytical stage, but he will need a different text to obtain any facility in the use of the tool for which his interest has been stimulated by this introduction. Such books in this country as well as abroad, confirm my observations that mathematics is more and more coming to be recognized as the desirable, yea, the necessary language of science, not only to physicists, chemists, and engineers, but to entomologists, horticulturists, zoologists, etc.

Perhaps Bertrand Russell's definition of mathematics as "the science in which we never know what we are talking about, nor what we say is true," in being restricted to entities in natural science rather than the realm of generalizations (abstractions) will lead to more and more true statements of facts. The author evidently prefers Whitehead's conception of mathematics as "the intellectual instrument which has been created for rendering clear the aspects of the world quantitatively."

H. L. SLOBIN

Begriff und Anwendungen des Differentials. By Alwin Walther. B. G. Teubner, Berlin, 1929. iv+96 pages.

This pamphlet, "The Concept and Applications of the Differential," is also the outgrowth of a series of lectures delivered by the author at Göttingen, in 1926. Pages 1–11 are in the nature of an introduction and give an historical and analytical setting of the development of the differential and increment of functions. Pages 11–17 deal with continuity and infinitesimals. Pages 17–45 deal with the derivative and differential. Pages 45–86 deal with the theorem of the mean, Taylor's formula, and the definite integral.

The pamphlet is in some respects a portion of the volume reviewed above and in other respects a further development of that portion. It really is a bit of the theory of functions, primarily the part that pertains to the differential calculus, and belongs in an introduction to a second course in calculus. Though the author again addresses himself to teachers of the natural sciences, in the secondary schools, this work, unlike the volume above, has primarily the point of view of the mathematician. It is probably of some value in bridging the gap between a first course of calculus and function theory.

H. L. SLOBIN

College Algebra. By N. R. Wilson and L. A. H. Warren. Oxford University Press, American Branch, New York, 1928. $2.08.

Many teachers who were reluctantly forced to the conclusion that "Hall and Knight" would not do for American colleges will be glad to consider this college algebra by two Canadians. In it an effort has been made to adapt the traditional English course to the needs of the American student.

Great pains have been taken in the presentation, particularly in the first part of the book. This has in some cases resulted in explanations so long as to

repel the student for whose benefit they were intended. Examples might be cited from the eleven rules of the first chapter, or the four page explanation of the characteristic in the sixth chapter. Space saved here and from the 27 pages of ratio and variation might have been used to supplement the very inadequate review that precedes the first chapter. In the later chapters the style is marred by a tendency to use the equality sign of an equation as the principal verb of the sentence. As a whole, however, the book is well written; the treatment of such topics as the progressions, permutations and combinations, the binomial theorem and mathematical induction compares very favorably with that of most of our texts.

Apart from its attention to presentation, the Americanization of this book takes such forms as the appearance of an index, a greater variety in the printing, the use of American money (even such sums as $mP); and, most important, the treatment of limits and convergence is made to agree with the usual practice in this country.

JOHN M. STETSON

A Simplified Presentation of Einstein's Unified Field Equations. By Tullio Levi-Civita. Translated by John Dougall. Blackie and Son, London, 1929. 22 pages.

This exposition can be recommended for those who have time to read only one paper on this subject. The formulas are given in such detail that the only prerequisites are an acquaintance with Maxwell's equations and a slight knowledge of general relativity. Only the mathematics of the theory is given. There is no discussion of the physical ideas, or of the probable place of the theory in physics.

K. W. LAMSON

Poetry and Mathematics. By Scott Buchanan. The John Day Co., New York, 1929. 197 pages.

The opening chapter gives the purpose of the book as being to show by use of Plato's dialectic method "what mathematics and poetry are and what they are not." The author sees the two as more or less successful attempts to deal with ideas. His denunciation of the professional mathematician or "prover of propositions," who through elaborate manipulation of formulae and figures succeeds in mystifying the student, is no more bitter than that of the professional in the poetry workshop who lays too great a stress on versification and prosody. In both cases the means or "vehicle" by which an idea is to be reached has been confused with the idea itself.

The first vehicle discussed is the mathematical figure as a symbol. There is a diverting section at this point in which the author, quoting the dialogue between the caterpillar and Alice in Lewis Carroll's "Alice in Wonderland," pictures the caterpillar as a typical mathematics teacher and Alice as both the

typical pupil and the figure to be understood. This he contends is a lesson in projective geometry. Another more serious example taken from Euclid's "Optics" is that of the figure in a moving projective plane with the point of convergence at the eye. These demonstrations show the figure in transformation just as a poem reveals the character in action. This comparison is more thoroughly developed further on in the book. The author gives at length a demonstration of the Pythagorean theorem and shows how the mathematical technique has there been used to define an abstract idea.

Next he turns to the field of numbers and discusses the two theories: that of operations where the entire number system is built up by starting with unity and applying different operators; and that of postulates, where the numbers are the elements in a system, and different relations being assigned to hold between them, the numbers are differentiated as reflexive, symmetric and transitive, or their opposites. Similar to the ordinal and cardinal numbers are the ordinal and cardinal elements in a story, the first where a series of events happen to a character and are reflected by him and the latter where the events come from the character and are transfigured by his habits and wishes.

From the discussion of space and number the author proceeds to the ratio, pointing out the linguistic bridge between poetry and mathematics built by the Greeks and Romans in their words "logos" and "ratio" or reason. This he now calls analogical thought and he traces the expansion of the analogy into the allegory, giving examples from old and modern literature. A corresponding mathematical example is Archimedes' principle of the lever developed in the fields of mechanics and astronomy. These ratios in combination give the algebraic equation. Trigonometry is a science built up on ratios and having these equations for its laws. Descartes developed his "geometrico-algebraic" allegory by the introduction of loci and equations. Just as the ratio varies in the algebraic equation, the dramatic character varies in the dramatic equation acting under conditions which determine the plot of the play.

Taking the ratio and letting the second term approach zero, we get the operator $\Delta y/\Delta x$ which, when applied to the algebraic equation gives the differential or fluxion. Here the idea of functionality is introduced, a new aspect of the mathematical object. Up to this point the similarity between poetry and mathematics has been stressed. The author now states that mathematics sees and deals with relations and has for its symbol the function, while poetry deals with qualities and the symbol it operates with is the adjective. The method of mathematics is analytic and that of poetry synthetic.

The relation between the esthetic object or symbol with which the two subjects work and the intellectual object or idea is that the former finds interpretation and comprehension of the latter. Symbols or metaphors are the analogies which upon expansion into the allegorical form and introduction of more adequate symbols approach the idea.

With the shift of attention from ratio and proportion to function the author sees a turning from mechanics to new routes to the absolute in which mathe-

matics and poetry are more closely bound. Modern Pythagoreans with their new analogies allow everything, numbers, forces, functions, etc., to vary, thus making mathematics correspond to the scientific observations. He gives an interesting picture of mathematics and poetry each developing as a series of concentric spheres, each expansion including the preceding; and says that as soon as relations can be found to hold between qualities a new sphere will be built including both series and we shall have an allegory which is mathematical and poetic. The tragedy is an example of such a sphere but ends in the inevitable destruction of the sphere. The comedy on the other hand uses the method of substitution applied in functional work such as series or equations in mathematics. Viewing the popularizer of science as the stage manager and ourselves as the actors, the author deplores the missing spirit of the play and the scarcity of clear and distinct ideas today.

MARGARET M. YOUNG

MATHEMATICS CLUBS

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CLUB TOPICS

1930 AS A CENTENNIAL YEAR IN THE HISTORY OF MATHEMATICS

By WALTER CROSBY EELLS, Stanford University, California

In continuation of previously published lists[1] of centennial dates in the history of mathematics, the following list of important 1930 centennial dates is presented.

B. C. 570. Birth of Anaximenes, prominent pupil of Thales, of Ionian school. Greek mathematician and astronomer.

470. Birth of Hippocrates of Chios, one of the greatest Greek geometers, celebrated for his studies of the quadrature of lunes.

370. Death of Democritus of Abdera, Thracian geometer.

A. D. 130. Appearance of Ptolemy's *Almagest*, standard work on mathematical astronomy for over one thousand years, until superseded by work of Copernicus and Kepler.

330. Death of Iamblicus, Alexandrian mathematician who advanced the Greek theory of numbers.

430. Death of the church father, St. Augustine, who discussed Zeno's celebrated paradoxes and arrived at a conception of the "actually infinite" as a constant.

530. Computation by the Hindoos of a table of sines for every $3\frac{3}{4}°$ of the quadrant.

1530. Probable date of birth of Bombelli, Italian mathematician, author of "the most teachable and the most systematic treatment of Algebra that had appeared in Italy up to that time" (Smith, D. E.: *Source Book in Mathematics*, p. 80) which was noteworthy for its treatment of cubic and biquadratic equations.

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[1] This Monthly, vol. 36 (1929) pp. 99-100 for a list of 1929 events and for references to previous volumes for corresponding lists from 1925 to 1928.

1530. Rudolff used a symbol of a vertical bar as a decimal point in a compound interest table.

1630. Birth of Barrow, English mathematician whose method of tangents was one of the forerunners of the differential calculus.

1630. Galileo first called attention to the cycloid and suggested that arches of bridges should be built in the form of this graceful curve.

1630. Galileo in his *Dialog über die beiden hauptsächlichtsen Weltsysteme* considers questions later recognized as belonging to the field of the calculus of variations.

1630. Publication of *Grammelogia, or the Mathematicall Ring*, by Delamain, mathematical teacher of London, earliest publication describing a slide rule. For facsimile of title page, with illustration of circular slide rule, see Smith, D. E., *Source Book in Mathematics*, p. 157.

1630. Death of Kepler, celebrated geometrician and astronomer, whose three laws form the basis of modern astronomy.

1730. Birth of Bezout, French mathematician, whose *Théorie générale des Equationes Algébriques* contained his theory of elimination and symmetric functions of the roots of an algebraic equation.

1730. Results of "Maclaurin's Theorem" first given by Stirling in his *Methodus Differentialis*.

1730. Publication of De Moivre's *Miscellanea Analytica*, which brought about his election to the Berlin Academy.

1730. Publication of English translation of Venema's *Coffer Konst*, in New York, second oldest arithmetic printed in America.

1830. Birth of Cremona, Italian geometer, discoverer of the "Cremona transformation," of fundamental importance in modern geometry.

1830. Publication of Legendre's monumental *Théorie des nombres* in two large quarto volumes (third, final, enlarged edition), still a standard work on number theory.

1830. Word "group" first used in the technical, mathematical sense by Galois.

1830. Death of Fourier, French physicist and mathematician, discoverer of "Fourier series" by which any arbitrary function of a real variable may be represented by a trigonometric series.

1830. Publication at Hartford of Barnard's *A Treatise on Arithmetic*, first American arithmetic to contain pictures to aid beginners in mastering the subject.

DISCORD IN MATHEMATICS LAND

By MARIE WHELAN, Hunter College of the City of New York

INTRODUCTORY SCENE: Two strips of white tape are laid out parallel on the floor. Math Student enters and walks along one of the lines saying slowly, wearily and rhythmically:
Math Student: "And *on* and *on* and *on* and *on* and *on*—(out of the room)."

SCENE I:

f(x): My children, we are gathered here in court assembled to consider matters grave.
From trusted sources it has reached mine ears
That discord rears its head in this our land.
The order that we cherish is assailed
And harmony is threatened in this its very home,
That harmony the like of which has ne'er been equalled in the world
Outside the borders of our realm.
And shall we tamely sit and watch it go?
It shall not be! On that I am resolved.
And so I've called you here today

To air your grievances in open court.
If any one has plaint against his brother
Let him speak now, or henceforth hold his peace.
Big *D* of *x*, take you your stand upon my right,
And differentiate between the various claimants for the floor,
And likewise calculate for us the rate of change from case to case,
That all may move with speed and order
As doth become this court.
And Integral of mine, do you be seated on my left
And lend attentive ear to every case,
So at the end to make summation for the court
Of all that does transpire.
Big *D*, proceed!

D_x: My Lord, Addition claims the floor.
He has a charge to make.

Addition: My Lord, I am a thrifty fellow, as you know,
And all my life have saved and saved and saved.
I have struggled hard and piled up wealth on wealth
And oft have been a multimillionaire.
I would not have a care or fear
For aught the future has in store
Save for the depredations of this thieving rogue, Subtraction.
Throughout the years, he's robbed me
Right and left, small sums and large
And often at one blow
Has stolen all my hard earned wealth away
And left me penniless.
Patiently have I endured till now
And countless times have struggled up
From want, to wealth exceeding great.
With what results? This rascal comes along
And steals it all away.
I am growing old and weary in the game.
I will not work with naught to show for it
Unless the court will guarantee my rights
And place restraint upon this thieving knave.
I have concluded.

$f(x)$: Subtraction, take the floor
And justify your actions if you can!

Subtraction: I've naught to say but this.
I claim the right to make a living
By means that nature has endowed me with
By taking here and there, and when and where
I can. A man must live, my Lord.

$f(x)$: My judgment I'll reserve upon this case.
Stand down! Big *D*, proceed!

D_x: Multiplication has the floor. I present him to the court.

Multiplication: Your Honor, and my Brothers, I've a charge against Division.
His malice brings to nothing all my toil.

My duty 'tis, and pleasure too,
To bring together diverse folk in this our land
And mould them into one,
And many happy unions have I formed
Which stood intact from all assaults save one
And that, Division's. He takes delight
In breaking every bond I make.
I'll cite a case in point.
I took *ab* and formed a union strong
With ρ' and with ρ.
The result was very happy. I was pleased
And they were pleased. And then
Division came along and spoiled my handiwork
And parted *ab* from his ρ's.
This instance is but one
Of thousands like to this.
His malice is unbounded, I maintain
My Lord will find he is the root and cause
Of all the discord that is rife today.
His very name proclaims it.
My grievances are many at his hands!

$f(x)$: Division! What say you to this charge?

Division: My Lord, his charge is foolish. Would you blame
The *light*, that it is seen,
Or *sound*, that it is heard,
Or *black*, that it is black,
Or *white*, that it is white?
His mission is to join.
My mission is to part.
And sometimes mine the greater blessing is.

$f(x)$: I cannot give decision on this case.
Who seeks the floor?

D_x: The Logarithm claims the floor.
I recognize his right.

Logarithm: My right and privilege it is
To go out walking by the side of x
And many happy hours have I spent
In this, my normal way of living.
And none there are that can contest my claim
Save only one, this base e, the Exponential.
Every time, with mischievous delight
He takes x from me. He has spoiled my life.
I ask the court to guarantee my rights
As prior unto his.

$f(x)$: Well, e, do you admit this claim?

e: Admit this claim? Why should I?
How does he rate a higher claim than mine?
When, as you know, without me

He never had been born. He has no rights,
Save those that I allow. I've been
Too generous with the fellow, that I see.
A fault I'll mend in future.

$f(x)$: It seems to me this is a case
Where x himself might have a word to say.
Let him be called before the court
To state his preference.

x: Which do I prefer? Why neither one, or both,
It all depends, my Lord.
You see, I like to go out walking with the Log
But, sometimes, I grow weary, and am quite content
To have the exponential rescue me.
For although a very little fellow, e is strong,
His custom is to bear me on his shoulders.
I find that very restful when I'm tired.

$f(x)$: Why then, it seems, 'tis e that you prefer.

x: Oh no, my Lord, not always.
I weary too of being carried thus
And signal to the Log to take me down.
I needs must have my exercise
Or soon would I grow old and stiff
And lose my famed agility,
And every stupid schoolboy catch me out
In all my tricks.
'T would never do, my Lord!

$f(x)$: Well, x, stand down! You do not help us much.
I cannot give my judgment, yet awhile.
We will hear the next case.

D_x: The Sine desires to speak.
The Sine, my Lord.

Sine: My Lord, I wish to bring complaint
Against the Inverse Sine.
The patient labor of my hands
He tears down, every time.
I speak not only for myself, but by request
For all my clan, the cosine and the rest.
They have the same complaint to make
Against the whole *arc*-tribe.
We've reached the limit——

(Excitedly, angrily, the Arc Sine interrupts.)

Arc Sine: He's reached the limit, I protest——

$f(x)$: (thundering) Silence! Who dares to speak
Without permission here?
I judge him guilty of contempt of court.
Let him forthwith be ejected,

To infinity projected,
By order of the Chair.

D_x: α, β, γ, δ, perform your office
And project the Arc Sine to infinity.

(α, β, γ, δ seize the Arc Sine and heave him out of the room)

$f(x)$: Well, Sine, you may continue now
Without fear of interruption.

Sine: I have nothing else to say, my Lord,
I simply wished to put our case before the court.
I am sure we shall have justice at your hands.

$f(x)$; Who speaks for the defendants in this case?

D_x: I recognize the Arc Cosine.

Arc Cosine: We are taken at a serious disadvantage
Our natural spokesman has been exiled
From the court—for cause, we do admit,
And must apologize that one of us
So far forgot the honor due the Chair.
But, truly, 'twas enough to overwhelm us all
To hear the very charges we prepared
Flung in our teeth. I, as impromptu spokesman for my group,
Do hurl them back again, where they belong.

D_x: The last case has been heard, my Lord.
There are no other claimants for the floor.

$f(x)$: Now, Integral of $f(x)$, do you perform
Your office for the court,
And sum the cases you have heard.
Be brief and to the point.

Integral of $f(x)$: Never have I had an easier summation
To perform. 'Tis briefly this:
That every citizen in our land
Is up in arms against his inverse neighbor.
My task is easy, yours I fear is hard.
No evidence was produced to fix the blame
On one side or the other.
It seems a very mutual affair.

$f(x)$: My task is difficult, I confess
I do not see my way, for every plaintiff
Seems as guilty or as free of blame
As the accused. It was my hope
To bring a common cause to view,
The root of all the trouble.
It has not so transpired. I need a light.

D_x: I think, perhaps, that I can help,
My Lord, if you will give me leave to speak.

$f(x)$: Speak up! Big D, if you have light to shed,
It never was more welcome.

D_x: My Lord, I am a discriminating fellow, as you know,
And I've been putting two and two together, all the day,
And now, at last, I have the answer—*four*.
Deep down into my own experience
I have dug, and hence derived
The common cause, the root of all our war,
The guilty culprit!
For every time that I attacked the work
Of my inverse ('tis true he did not charge me),
One there was who put me up to it.
I think, if you'll inquire, that you will find
The same is true of all my brothers.
Who is that one? Well known is she
To all of you—the Math Student!

(Excited cries from the court—several speaking at once)

Why yes, 'tis true! She did! She is the one!
'Tis true, 'tis true! She is to blame!

$f(x)$: Ha! It seems we have the truth at last.
Now, now I see my way.
Let the Math Student be summoned before me.

Math Student, it has been clearly shown
That you, and you alone
Have undermined the harmony of Math Land.
You are not one of us, but an intruder,
A harmless one, we thought,
Whom we have suffered to come and go
Amongst us. And how have you repaid us?
Nearly have you brought my realm about my ears
In civil strife. For justice' sake you must be punished.
Prepare yourself to hear the sentence of the court.
At your feet you see two tracks upon the ground.
Start walking now upon the right hand one,
And at the rate of two steps to the second—
Go on and on and on!
You may come back on the parallel track—
But don't turn round 'till they meet.
March on! You'll trouble us no more.

(The court files out of the room leaving Math Student who toes the tape line saying slowly, wearily, rythmically)

Math Student: And *on* and *on*, and *on* and *on*
And *on* and *on* I sped,
And *on* and *on*, and *on* and *on*
I'd still be going ahead,
But—the alarm went off (smiling and bowing)
At half past six
And woke me out of bed.

PROBLEMS AND SOLUTIONS

Edited by B. F. Finkel, Otto Dunkel, and H. L. Olson

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editor will be glad to assist members of the Association with their difficulties in the solution of such problems.

3412. *Proposed by P. R. Rider, Washington University.*

The vertex of a right circular cone of semi-vertical angle α is at the center of a unit cube, the axis of the cone coinciding with a diagonal of the cube. Find the area of that part of the conical surface which is inside the cube.

3413. *Proposed by P. R. Rider, Washington University.*

The axis of a right circular cylinder of radius r coincides with a diagonal of a unit cube. Find the area of that part of the cylindrical surface which is inside the cube.

3414. *Proposed by B. C. Wong, Berkeley, California.*

Prove or disprove:

$$\sum_{i=0}^{t}\left[\frac{(r-1)!(r-2i)}{(r-i)!i!}\right]^2=\frac{(2r-2)!}{r!(r-1)!},$$

where $t=(r-2)/2$ if r is even and $t=(r-1)/2$ if r is odd.

3415. *Proposed by L. S. Johnston, University of Detroit.*

Let A, C, O, C', A' be collinear points, with $OA=OA'$, $OC=OC'$; let l_1, l_2, and l_3 be lines perpendicular to AA' through A, O, and A' respectively. With any point P on l_2 as center and PC as radius a circle is drawn. Let d_1 and d_2 be the diameters determined by the intersections of this circle with l_1 and l_3. Find the envelope of d_1 and d_2.

3416. *Proposed by William Hoover, Columbus, Ohio.*

If a circumscribed triangle to a given circle have two sides fixed and the third variable, to determine the envelope of its circumcircle.

3417. *Proposed by C. O. Williamson, The College of Wooster, Wooster, Ohio.*

Construct a square so that each side shall pass through a given point.

3418. *Proposed by David H. Dodge, Los Angeles, California.*

Prove that

$$\frac{7\sum n^6 + 5(p+1)\sum n^4 + p\sum n^2}{7\sum n^6 - 5(p-1)\sum n^4 - p\sum n^2} = \frac{n^2+n+p}{n^2+n-p}.$$

where

$$\sum n^6 = (6n^7 + 21n^6 + 21n^5 - 7n^3 + n)/42\ ;$$
$$\sum n^4 = (6n^5 + 15n^4 + 10n^3 - n)/30\ ;$$

and

$$\sum n^2 = (2n^3 + 3n^2 + n)/6.$$

SOLUTIONS

3028 [1923, 275]. *Proposed by Norman Anning, University of Michigan.*

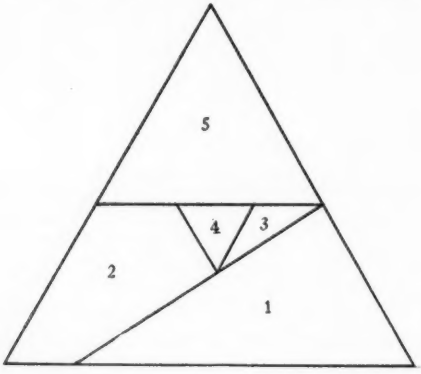
Equilateral triangles are described on the sides of a right triangle. Dissect the triangles on the legs and reassemble the parts to form the triangle on the hypotenuse.

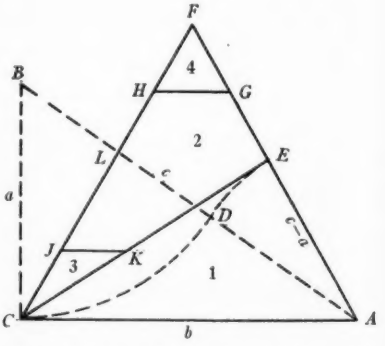
3048 [1923, 450]. *Proposed by H. C. Bradley, Massachusetts Institute of Technology.*

Cut two equilateral triangles of any relative proportions into not more than five pieces which can be assembled to form a single equilateral triangle.

Solution by H. C. Bradley, Massachusetts Institute of Technology.

The joker in this problem lies in the fact that one of the given triangles is not cut at all. Let a and b be the sides of the given two equilateral triangles,





$b \geqq a$. Form the right angled triangle ABC with the right angle at C, $a=BC$, $b=CA$, and $AB=c$. On BA lay off $BD=BC=a$. Let F be the third vertex of the equilateral triangle FCA, and lay off on AF the length $AE=AD=c-a$. Bisect EF in G and lay off on CF the lengths CJ and HF, each equal to FG. Draw CE; also draw JK parallel to CA cutting CE in K, which will lie within the segment CE. Then draw HG. The triangle FCA may be cut into four pieces, CAE, CKJ, $JKEGH$, HGF, which may be assembled to form a trape-

zoid with the equal slant sides AE, JH, and with parallel sides of lengths $c=JK+CA$ and $a=2HG+JK$. Upon this trapezoid the equilateral triangle of side a may be placed so as to complete the equilateral triangle of side c.

No piece is turned over, and only one piece need be rotated. This construction is always possible leaving intact the smaller, or equal, triangle of side a. It is also possible to leave the triangle of side b intact and to cut up the other with the side a in the same manner, provided $a>\frac{3}{4}b$. For the special case $a=\frac{3}{4}b$ there are also two solutions making only four pieces. We may divide a by two straight line cuts parallel to the base into three pieces having equal altitudes; or we may divide b into three pieces by two straight line cuts parallel to the sides, one of length $\frac{1}{2}b$ and the other of length $\frac{1}{4}b$.

The proof of the construction is simple. It may be shown that $JK=c-b$ by comparing the triangle CKJ with the similar triangle having a vertex at C and a base through E parallel to JK.

There are other special cases, one of which is given by $3^{1/2}a=b$. Here each triangle may be split into two equal right triangles, and these four pieces may be reassembled to form an equilateral triangle. Or a may be left intact, while b is divided first into two equal right triangles, and then one of these parts is divided into an isosceles triangle of base b and a right triangle of altitude $\frac{1}{2}b$.

3373 [1929, 232]. *Proposed by Paul Capron, U. S. Naval Academy.*

A plane cuts the lateral edges of a quadrangular pyramid so that the ratios of the edges of the whole pyramid to the corresponding edges of the smaller pyramid cut off are x_1, x_2, x_3, x_4 and the ratio of the smaller volume to the larger is r. Either diagonal of the base of the larger pyramid divides this base into triangles whose areas bear the ratio r_i, r_{i+2} to the whole area of the base, each of these ratios being numbered to correspond with the lateral edge opposite its triangle, so that $r_1+r_3=1=r_2+r_4$. Show that

$$x_1x_2x_3x_4r = x_1r_1 + x_3r_3 = x_2r_2 + x_4r_4.$$

Solution by J. H. Neelley, Carnegie Institute of Technology.

Let E_i and e_i, $i=1, 2, 3, 4$ be the edges of the large and small pyramids respectively. Then the plane through E_2 and E_4 divides the large pyramid into two triangular pyramids of volumes P_1, P_3 and the small pyramid into two triangular pyramids of volumes p_1, p_3. Now $P_1\cdot p_1^{-1}=x_2x_3x_4$ and $P_3\cdot p_3^{-1}=x_1x_2x_4$, being pairs of triangular pyramids with a triedral angle equal. Therefore

$$p_1 + p_3 = P_1(x_2x_3x_4)^{-1} + P_3(x_1x_2x_4)^{-1};$$

and dividing this equation by P, the whole pyramid, and using

$$(p_1 + p_3)\cdot P^{-1} = r, \quad P_1\cdot P^{-1} = r_1, \quad P_3\cdot P^{-1} = r_3,$$

we have when we clear of fractions:

$$x_1x_2x_3x_4r = x_1r_1 + x_3r_3.$$

Similarly, using the plane through E_1 and E_3, we get the second relation.

Also solved by A. Pelletier and B. D. Roberts.

3374 [1929, 233]. *Proposed by J. Rosenbaum, Milford, Conn.*

Given two triangles, one within the other, to construct a third triangle which shall be inscribed in the outer and circumscribed about the inner triangle. Also prove that if the two given triangles are equilateral and concentric the third triangle is equilateral.

Solution by Philip Franklin, Massachusetts Institute of Technology.

Let $A'B'C'$ be a triangle inside ABC. Take P any point on AB, draw straight line $PA'Q$ to cut BC in Q; $QB'R$ to cut CA in R and $RC'S$ to cut AB in S. As P varies, its range is perspective to that of Q, and hence projective to it, similarly through B' to the range of R and finally through C' to that of S. If three points P are taken, and the corresponding S points are found, we may find the double points of this projectivity by the standard construction (cf. Veblen and Young, Projective geometry, vol. I., p. 246) and so locate the two points which are the vertices of triangles of the kind wanted. With a given order, $A'B'C'$, there are two solutions, one solution, or no solution according to the number of real distinct double points; with this order not given, there are not more than twelve solutions. Of course the vertices are on the sides produced in some of these cases. (cf. Veblen and Young, l. c. problem 4, p. 250, and the reference there).

When both triangles are equilateral and concentric, and so lettered that a rotation of 120° about the center taking ABC into BCA takes $A'B'C'$ into $B'C'A'$, the two triangles found above will be equilateral, and concentric with the original pair. For, let their sides be pqr, and $p'q'r'$. Suppose that a rotation of 120° carries p, q, r into q', r', p'; then the same rotation must carry p', q', r' into q, r, p. A further rotation of 120° in the same sense must then carry q', r', p' into r, p, q; and a third rotation of 120° carries r, p, q into p', q', r'. Since a rotation of 360° makes p, q, r fall on p, q, r; p, q, r and p', q', r' coincide and there is only one solution, and it must be equilateral. If there are two distinct solutions, a rotation of 120° must carry p, q, r into q, r, p.

Also solved by the proposer.

3377 [1929, 285]. *Proposed by T. S. Peterson, The Ohio State University.*

Evaluate the summation

$$\sum_{i=0}^{n} \frac{(2i+1)!(2n-2i+1)!}{i!(i+1)!(n-i)!(n-i+2)!}.$$

Solution by B. P. Gill, College of the City of New York.

Writing the above expression in the form

$$f(n) = \sum_{i=0}^{n} \binom{2i+1}{i} \binom{2n-2i+1}{n-i} \cdot \frac{1}{n-i+2}, \tag{1}$$

we prove by induction that

$$f(n) = \frac{1}{2}\binom{2n+3}{n}. \tag{2}$$

Changing i in (1) to $n-i$, and adding the result to (1), we get

$$2f(n) = \sum_{i=0}^{n} \binom{2i+1}{i}\binom{2n-2i+1}{n-i}\cdot\frac{n+4}{(n-i+2)(i+2)}. \tag{3}$$

From (1),

$$f(n+1) = \sum_{i=0}^{n+1} \binom{2i+1}{i}\binom{2n-2i+3}{n-i+1}\cdot\frac{1}{n-i+3}$$

$$= \frac{1}{n+3}\binom{2n+3}{n+1} + \sum_{i=1}^{n+1} \binom{2i+1}{i}\binom{2n-2i+3}{n-i+1}\cdot\frac{1}{n-i+3}.$$

In the second term on the right, we note that

$$\binom{2i+1}{i} = \binom{2i-1}{i-1}\cdot\frac{2i(2i+1)}{i(i+1)} = \binom{2i-1}{i-1}\left(4 - \frac{2}{i+1}\right).$$

Then changing i to $i+1$, and using (1) and (3), we get

$$f(n+1) = \frac{1}{n+3}\binom{2n+3}{n+1} + 4f(n) - \frac{4f(n)}{n+4}$$

$$= \frac{1}{n+3}\binom{2n+3}{n+1} + \frac{4(n+3)f(n)}{n+4}.$$

With the initial values $f(0) = 1/2$, $f(1) = 5/2$, the induction is completed by substituting in this recurrence relation from (2) and carrying out some easy algebraic reductions.

3378 [1929, 285]. *Proposed by Paul Wernicke, Washington, D. C.*

If a tetrahedron $A = A_1A_2A_3A_4$ has altitudes concurrent (in an orthocenter), then, denoting by (ij) the angle at A_j of the triangular face a_i (opposite A_i), the following products are equal: tan (41)·tan (32)·tan (23)· tan (14) = tan (31)·tan (42)·tan (13)·tan (24) = tan (21)·tan (12)·tan (43)·tan (34). The factors of each product are tangents of the four plane angles made by the edges which form a tetragram.

Solution by A. Pelletier, Montreal, Canada.

Let O be the orthocenter. The three planes passing through A_iO and the three edges concurring in A_i determine the three altitudes of the face a_i. Calling P_{jk} the intersection of the plane through A_iA_l with A_jA_k, we have:

$$\tan (ij) = \frac{A_iP_{jk}}{A_jP_{jk}} = \frac{A_kP_{ji}}{A_jP_{ji}}.$$

Hence

$$\tan (31) \tan (42) \tan (13) \tan (24) = \frac{A_2P_{14}}{A_1P_{14}} \frac{A_1P_{23}}{A_2P_{23}} \frac{A_4P_{23}}{A_3P_{23}} \frac{A_3P_{14}}{A_4P_{14}},$$

$$= \frac{A_3P_{14}}{A_1P_{14}} \frac{A_4P_{23}}{A_2P_{23}} \frac{A_1P_{23}}{A_3P_{23}} \frac{A_2P_{14}}{A_4P_{14}},$$

$$= \tan (21) \tan (12) \tan (43) \tan (34).$$

Interchanging 2 and 4 in the above, we obtain:

$$\tan (31) \tan (24) \tan (13) \tan (42) = \tan (41) \tan (14) \tan (23) \tan (32).$$

3379 [1929,285] *Proposed by J. H. Neelley and T. L. Smith, Carnegie Institute of Technology.*

Two men own jointly x cows which they sell for x dollars per head and with the returns buy sheep at \$12 per head. As their income from the cows is not divisible by 12 they purchase a lamb with the remainder. Later they divided the flock so that each had the same number of animals. How much money was due the man with the lamb by the other man?

Solution by P. S. Ganesa Sastri, Trichinopoly, S. India.

The price of x cows at x dollars a head is x^2 dollars. Since x^2 is not exactly divisible by 12, x cannot be 6 or any multiple of 6. Hence it should be $12n \pm y$, where n is any integer and y is an integer ranging from 1 to 5.

Now $(12n \pm y)^2$ when divided by 12 should leave a remainder and have an *odd* integer for its quotient.

$$(12n \pm y)^2 \div 12 = (144n^2 \pm 24ny + y^2) \div 12$$

$$= 12n^2 \pm 2ny + \frac{y^2}{12}.$$

But $12n^2 \pm 2ny$ is always even; therefore $y^2/12$ should leave a remainder and have an *odd* integer for its quotient. Since y is an integer ranging from 1 to 5, y must be 4. Since the remainder when 4^2 is divided by 12 is 4, the cost of the lamb is 4 dollars. Therefore the man who has one sheep more should pay the man having the lamb 4 dollars, the number of cows being $12n \pm 4$.

Also solved by W. E. Buker, P. S. Dwyer, C. H. Davis, Marie M. Johnson, H. S. Kaltenborn, A. Pelletier, E. A. Whiteman, and Margaret M. Young.

3381 [1929, 286]. *Proposed by E. B. Escott, Oak Park, Ill.*

The following construction is taken from a book on mechanical drawing: *Problem.* To bisect the angle between two given lines whose point of intersection is beyond the limits of the drawing.

Construction. Let the given lines be CD and AB. With an arbitrary point E on CD as center and any radius strike an arc cutting CD in I and AB in G. With G as center and the same radius strike an arc cutting AB in L and CD in E. Draw the common chord of the two circles, HF. With I and L as centers and the same radius strike arcs cutting in J and K. Draw JK. The intersection of HF and JK is O, a point on the bisector.

To find a second point, with O as center, strike an arc cutting AB and CD in two points M and N. With M and N as centers and the same radius strike arcs cutting in P. Draw OP, the required bisector.

Give the proof of the construction.

Solution by Margaret M. Young, Hunter College of the City of New York.

Since O lies on the common chord of the equal circles with centers E and G, $OE=OG$. O also lies on the common chord of the equal circles with centers I and L and hence $OI=OL$. By construction $EI=LG$ and it follows that the triangles EOI and OGL are congruent. Hence their attitudes are equal and O lies on the bisector of one of the angles between AB and CD. Let V be the intersection of AB and CD. Then, since VO bisects the angle under consideration and $ON=OM$, we have $VN=VM$, if M and N have been properly chosen. Hence VO is the perpendicular bisector of MN; and, since $NP=MP$, P must also lie on VO produced.

Note by the Editors. The essentials of this construction are as follows: Two equal segments EI and LG are laid off on the lines CD and AB, respectively. Then the perpendicular bisectors of EG and IL meet on the bisector of one of the angles between CD and AB, while the perpendicular bisectors of EL and IG meet on the bisector of the other angle.

Also solved by A. Pelletier.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

The Third International Congress of Applied Mathematics will be held at Stockholm, August 24–29, 1930.

On the occasion of the opening of its new chemical laboratory, Princeton University conferred honorary doctorates on Irving Langmuir, of the General Electric Company, and Jean Baptiste Perrin, director of the laboratory of physical chemistry at the University of Paris.

The division of mathematics at Harvard University has established a new mathematical society called the Harvard Mathematical Colloquium. The meetings occur fortnightly, and are open to all mathematicians in Cambridge and its vicinity. Among the speakers of the year are the following: Professors

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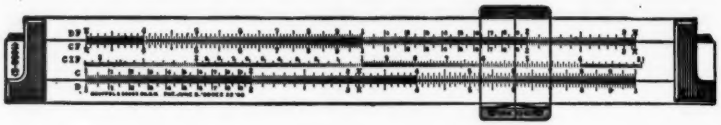

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MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fourteenth Summer Meeting of the Association, Providence, Rhode Island, Sept. 8-9, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1930.

ILLINOIS, Lake Forest, Ill., May 2-3.
INDIANA, Earlham College, May 2-3.
IOWA, May 2-3.
KANSAS, February 15.
KENTUCKY, Lexington, Ky., April 5.
LOUISIANA-MISSISSIPPI, Cleveland, Miss., March 7-8.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA.
MICHIGAN, Ann Arbor, Mich., March 22.
MINNESOTA, Carleton College, May 17.
MISSOURI.
NEBRASKA, Peru, Neb., May 9.
OHIO, Columbus, Ohio, April 3.
PHILADELPHIA, Philadelphia, Pa., November 29.
ROCKY MOUNTAIN, Denver, Colo., April 11-12.
SOUTHEASTERN.
SOUTHERN CALIFORNIA, University of Southern California, Los Angeles, Calif., March 8.
TEXAS.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

Graustein, Rado, Birkhoff, Walsh, Morse, and Stone of Harvard, Professors Wiener, Struik, and Franklin of the Massachusetts Institute of Technology, and Messrs, Morrey and Currier of Harvard.

The Mathematical Association of America has awarded its Chauvenet prize for mathematical exposition to Professor T. H. Hildebrandt, of the University of Michigan, for his paper entitled *The Borel theorem and its generalizations*, published in this Bulletin vol. 32 (1926), pp. 423–474.

Professor G. N. Lewis, of the Universtiy of California, has been elected an honorary member of the Swedish Academy of Sciences. Professor Lewis has also recently received an honorary doctorate from the University of Chicago, for his contributions to physical chemistry.

The following have been appointed to National Research Fellowships in mathematics for 1929–30 (this list includes reappointments): A. C. Berry, A. B. Brown, L. W. Cohen, C. C. Craig, Jesse Douglas, H. T. Engstrom, J. J. Gergen, B. W. Jones, N. H. McCoy, Morris Marden, Gordon Pall, W. T. Reid, J. H. Roberts, N. W. Rutt, Marie J. Weiss.

Mr. B. R. Beisel, of Cornell, has been appointed assistant professor of mathematics at the Uniontown Center of the University of Pittsburgh.

Professor G. A. Bliss, of the University of Chicago, delivered a lecture on "The mathematics of the new quantum theory" at Carleton College on Feb. 17 and at the University of Minnesota on Feb. 18.

Mr. H. V. Craig, of the University of Wisconsin, has been appointed adjunct professor of mathematics at the University of Texas.

Professor R. E. Gilman, of Brown University, will be the associate editor in charge of the department of "Questions and Discussions" of the MONTHLY beginning with the April issue.

Associate Professor J. W. Harrell, of Baylor University, Waco, Texas, has been promoted to a professorship and head of the department of Mathematics. He is a charter member of the Mathematical Association.

Dr. Asaph Hall, of the United States Naval Observatory, who retired from a professorship in the Navy in June, 1929, died on January 12, 1930, in his seventy first year.

Professor E. A. Pattengill, of Iowa State College, Ames, Iowa, died on February 10, 1930, at the age of 55 years.

MATHEMATICS AND THE PROBLEM OF ORE LOCATION[1]

By WARREN WEAVER, University of Wisconsin

A. Introduction

1. *Earth physics.* A large part of modern science consists of seeing the unseen. Indeed it is the essence of science to penetrate behind those external phenomena which appeal directly to our senses, and discover those inner relationships which are visible only to the eye of the mind. Thus today we probe into the atom and the cell, and we reach far out to the extra-galactic nebulae, always coordinating and interpreting the unseen and unseeable in terms of the seen. Recent popular expositions have, to a considerable extent, acquainted the public mind with the two most dramatic aspects of modern physics,—atomic theory, as it reveals to us the structure and behavior of the unimaginably small, and astro-physics, as it reveals to us the constitution and history of the unimaginably vast.

About midway between these two, as regards the scale of dimensions involved, is a third important division—that of earth physics. As old or older than either atomic or astro-physics, it has perhaps been less vigorously developed of late since it is not much concerned with the quantum and relativity theories. Earth physics, in fact, is concerned with the application of the older theoretical subjects such as classical electromagnetic theory, potential theory, and the theory of the propagation of waves in continuous elastic media.

There are, of course, very many branches and sub-divisions of the great science of earth physics. In its broadest aspect it articulates with astro-physics. In its most fundamental aspect it articulates with molecular physics. Between these two limits is a vast range of subject matter. Thus earth physics is concerned with the great problems of the origin and age of the earth; with the thermal history of the earth's surface and interior; with land and sea tides and their friction effect upon evolution of form and motion; with the theory of isostasy and its application to surface features; with the hydrodynamics of underground and surface waters; with the mechanics of plastic flow and elastic deformation as these furnish powerful investigative tools in structural geology; with meteorology in all of its manifold aspects; with terrestrial magnetism; w th atmospheric electricity; and with many, more specialized, subjects. I have taken the opportunity to mention these challenging and significant fields of research, even though we have no concern here with them, in order to register thereby a mild protest against the notion, apparently held by some, that the only worth while fields of modern physical research are those which concern either an atom or the cosmos. Here, as in all matters, there is a middle ground.

It has been noted above that a great purpose of science is to see with the aid of experiment and of the mind's eye what is otherwise unseen; for it is only

[1] Read at the Annual meeting of the Mathematical Association of America at Des Moines, Iowa on Jan. 1, 1930.